

UMBILIC POINTS AND REAL HYPERQUADRICS

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ABSTRACT. There exist polynomial identities asociated to normal form, which yield an existence and uniqueness theorem. The space of normalized real hypersurfaces has a natural group action. Umbilic point is defined via normal form. A nondegenerate analytic real hypersurface is locally biholomorphic to a real hyperquadric if and only if every point of the real hypersurface is umbilic.

0. Introduction

An analytic real hypersurface M is said to be in Chern-Moser normal form if M is defined by the following equation near the origin:

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where

$$\langle z, z \rangle \equiv z^1 \bar{z}^1 + \cdots + z^e \bar{z}^e - z^{e+1} \bar{z}^{e+1} - \cdots - z^n \bar{z}^n$$

for a positive integer e in $\frac{n}{2} \leq e \leq n$, and

$$F_{st}(\mu z, \nu \bar{z}, u) = \mu^s \nu^t F_{st}(z, \bar{z}, u)$$

for all complex numbers μ, ν , and the functions F_{22}, F_{23}, F_{33} satisfy the condition

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0.$$

Here the operator Δ is defined as follows:

$$\Delta \equiv D_1 \bar{D}_1 + \cdots + D_e \bar{D}_e - D_{e+1} \bar{D}_{e+1} - \cdots - D_n \bar{D}_n,$$

$$D_k = \frac{\partial}{\partial z^k}, \quad \bar{D}_k = \frac{\partial}{\partial \bar{z}^k}, \quad k = 1, \dots, n.$$

Then we have an existence theorem of a biholomorphic normalizing mapping(cf. [CM], [Pa2]).

Theorem 0.1 (Chern-Moser). *Let M be an analytic real hypersurface with nondegenerate Levi form at the origin in \mathbb{C}^{n+1} defined by the following equation:*

$$v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0.$$

Then there exists a biholomorphic normalizing mapping ϕ of M such that $\phi(M)$ is in Chern-Moser normal form.

We have a uniqueness theorem of a biholomorphic normalizing mapping(cf. [CM], [Pa2]).

Theorem 0.2 (Chern-Moser). *Let M be the real hypersurface in Theorem 0.1 and $\phi = (f, g)$ in $\mathbb{C}^n \times \mathbb{C}$ be a biholomorphic normalizing mapping of M into Chern-Moser normal form such that*

$$f|_0 = g|_0 = 0.$$

Then ϕ is uniquely determined by the initial value $\sigma = (C, a, \rho, r)$ given by

$$C = \left. \frac{\partial f}{\partial z} \right|_0, \quad -Ca = \left. \frac{\partial f}{\partial w} \right|_0, \quad \rho = \Re \left(\left. \frac{\partial g}{\partial w} \right|_0 \right), \quad 2\rho r = \Re \left(\left. \frac{\partial^2 g}{\partial w^2} \right|_0 \right).$$

Let M be an analytic real hypersurface defined by the equation

$$v = \langle z, z \rangle + \sum_{k=3}^{\infty} F_k(z, \bar{z}, u)$$

and ϕ be a (possibly formal) biholomorphic normalizing mapping with identity initial value in $\mathbb{C}^n \times \mathbb{C}$ such that

$$\phi = \left(z + \sum_{k=2}^{\infty} f_k(z, w), w + \sum_{k=3}^{\infty} g_k(z, w) \right).$$

Suppose that the real hypersurface $\phi(M)$ is defined by the (possibly formal) equation

$$v = \langle z, z \rangle + \sum_{k=3}^{\infty} F_k^*(z, \bar{z}, u).$$

Then we obtain a following family of polynomial identities(cf. [CM]):

$$(0.1) \quad \begin{aligned} & \Re\{2\langle z, f_{m-1}(z, w) \rangle + ig_m(z, w)\}|_{w=u+i\langle z, z \rangle} \\ &= F_m(z, \bar{z}, u) - F_m^*(z, \bar{z}, u) + R_m(z, \bar{z}, u) \end{aligned} \quad \text{for } m \geq 3,$$

where $R_m(z, \bar{z}, u)$ is a polynomial of weight m consisting of the functions

$$f_{k-1}, \quad g_k, \quad F_k, \quad F_k^*, \quad k \leq m-1.$$

In the thesis [Pa1], we studied the polynomial identity (0.1) in each weight in order to investigate the process of determining the normalizing mapping ϕ and the normalized real hypersurface $\phi(M)$. Under accepting Theorem 0.1 and Theorem 0.2 to be proven independently(cf. [CM], [Pa2]), we obtain the following existence and uniqueness theorem from the polynomial identities (0.1):

Theorem 0.3. *There is a natural isomorphism for each $k \geq 3$ via the polynomial identity (0.1) such that*

$$\{F_l : l \leq k\} \simeq \{(f_{l-1}, g_l) : l \leq k\} \oplus \{F_l^* : l \leq k\}.$$

Hence there exist two unique mappings for each $k \geq 3$ such that

$$\begin{aligned}\nu &: \{F_l : l \leq k\} \longmapsto \{(f_{l-1}, g_l) : l \leq k\} \\ \kappa &: \{F_l : l \leq k\} \longmapsto \{F_l^* : l \leq k\}.\end{aligned}$$

Further, the formal biholomorphic mapping

$$\phi = \left(z + \sum_{k=2}^{\infty} f_k(z, w), w + \sum_{k=3}^{\infty} g_k(z, w) \right)$$

is convergent so that $\phi(M)$ is an analytic real hypersurface defined by the equation

$$v = \langle z, z \rangle + \sum_{k=4}^{\infty} F_k^*(z, \bar{z}, u).$$

Kruzhilin [Kr] showed that some low order terms of a normalization ϕ of M are equal to the low order terms of a local automorphism of a real hyperquadric whenever the real hypersurface M is already in normal form. We present a simple proof of Kruzhilin's Lemma. Then, as its consequence, we show that there is a natural group action on the space of analytic real hypersurfaces in normal form by the isotropy group H of a real hyperquadric via normalizations.

E. Cartan [Ca] and Chern-Moser [CM] have introduced umbilic points as a local CR invariant in their geometric theory and Chern-Moser identified the condition in normal form so that, on a nondegenerate analytic real hypersurface M , a point $p \in M$ is umbilic if there is a normal coordinate with center at $p \in M$ such that, for $\dim M = 3$,

$$(0.2) \quad v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2, \max(s,t) \geq 4} F_{st}(z, \bar{z}, u)$$

where

$$F_{24}(z, \bar{z}, 0) = F_{42}(z, \bar{z}, 0) = 0$$

and, for $\dim M \geq 5$,

$$(0.3) \quad v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where

$$F_{22}(z, \bar{z}, 0) = 0.$$

We show that this condition of umbilic points in normal coordinate may be taken to be the definition of umbilic points. Then we shall prove our main theorem in this article on umbilic points and real hyperquadrics.

Theorem 0.4. *Let M be an analytic real hypersurface with nondegenerate Levi form and U be an open neighborhood of a point $p \in M$ such that $U \cap M$ consists of umbilic points. Then the open subset $U \cap M$ is locally biholomorphic to a real hyperquadric.*

We may view Theorem 0.4 as an analytic analogue of E. Cartan's equivalence problem of spherical real hypersurfaces, which concerns the local existence of biholomorphic mapping of a spherical real hypersurface M to a real hyperquadric(cf. [Ca], [CM]). In the case that the Levi form on M is definite. Then Theorem 0.4 assures the local existence of biholomorphic mapping of a spherical real hypersurface M to a sphere S^{2n+1} .

On a nondegenerate analytic real hypersurface M , a point $p \in M$ is called spherical if there exist a neighborhood U of p and a biholomorphic mapping ϕ on U such that

$$\phi(U \cap M) \subset Q$$

where Q is a real hyperquadric. Then Theorem 0.4 is just a characterization of a spherical point p by umbilic points near p .

1. EXISTENCE AND UNIQUENESS THEOREM

I. In this article, we concern local properties of real hypersurfaces under biholomorphic mappings so that each real hypersurface has a distinguished point as its base point, i.e., the origin in \mathbb{C}^{n+1} , and each biholomorphic mapping leaves the origin invariant unless specified otherwise.

Once we agree to define a real hypersurface M locally by an equation of the form

$$v = F(z, \bar{z}, u).$$

Then there is one-to-one correspondence between the real hypersurface M and the defining equation. Hence we may identify the real hypersurface M locally with the defining equation unless serious confusion. We may abuse notations in this regard.

We shall study some consequences of Theorem 0.1 and Theorem 0.2.

Lemma 1.1 (Chern-Moser). *Let M be a nondegenerate analytic real hypersurface defined by*

$$v = F(z, \bar{z}, u) \equiv \sum_{k=2}^{\infty} F_k(z, \bar{z}, u).$$

Let $\phi = (\sum_k f_k, \sum_k g_k)$ be a normalization of M with initial value (C, a, ρ, r) such that $\phi(M)$ is defined by the equation

$$v = \langle z, z \rangle + \sum_{k=4}^{\infty} F_k^*(z, \bar{z}, u).$$

Then there is a family of polynomial identities, for each $k \geq 3$,

$$\mathcal{L}_k(f_{k-1}, g_k, F_k^*) = \rho F_k(z, \bar{z}, u) + R_k(z, \bar{z}, u)$$

where $R_k(z, \bar{z}, u)$, $k \geq 3$, is represented by a finite linear combination of finite multiples of the functions

$$f_{s-1}, \quad g_s, \quad F_s, \quad F_s^*, \quad s \leq k-1, \quad \text{and their derivatives.}$$

Proof. By Theorem 0.1, there is a biholomorphic normalization of M , $\phi = (f, g)$. Then we obtain the following identity

$$\begin{aligned} & \Im g(z, u + iF(z, \bar{z}, u)) \\ &= \langle f(z, u + iF(z, \bar{z}, u)), f(z, u + iF(z, \bar{z}, u)) \rangle \\ (1.1) \quad & + F^*(f(z, u + iF(z, \bar{z}, u)), \overline{f(z, u + iF(z, \bar{z}, u))}, \Re g(z, u + iF(z, \bar{z}, u))). \end{aligned}$$

We expand the identity with respect to weight and collect terms of the same weight so that

(1) for weight 1 and 2,

$$(1.2) \quad \begin{aligned} \Im g_1(z, 0) &= 0 \\ \Im g_2(z, u + iF_2(z, \bar{z}, u)) &= \langle f_1(z, 0), f_1(z, 0) \rangle \end{aligned}$$

(2) for weight $m \geq 3$,

$$\begin{aligned} &\Im g_m(z, u + iF_2(z, \bar{z}, u)) - \Re g_2(0, 1)F_m(z, \bar{z}, u) \\ &- 2\Re \langle f_1(z, 0), f_{m-1}(z, u + iF_2(z, \bar{z}, u)) \rangle \\ &- F_m^*(f_1(z, 0), \overline{f_1(z, 0)}, \Re g_2(z, u + iF_2(z, \bar{z}, u))) \\ &= R_m(z, \bar{z}, u) \end{aligned}$$

where the polynomial $R_m(z, \bar{z}, u)$ is given by a finite linear combination of finite multiples of the following functions

$$f_{s-1}, \quad g_s, \quad F_s, \quad F_s^*, \quad s \leq m-1, \quad \text{and their derivatives.}$$

With the expansion

$$g_2(z, u + iF_2(z, \bar{z}, u)) = g_2(z, 0) + g_2(0, 1)u + ig_2(0, 1)F_2(z, \bar{z}, u),$$

in the equality (1.2), we obtain

$$\langle f_1(z, 0), f_1(z, 0) \rangle = \Re g_2(0, 1) \sum_{\alpha, \beta} \left(\frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} \Big|_0 \right) z^\alpha \bar{z}^\beta$$

so that

$$g_2(z, w) = \Re g_2(0, 1) \times \{1 - iF_2(0, 0, 1)\} \{w - iF_2(z, 0, 0)\}.$$

Note that $\Re g_2(0, 1) \neq 0$ necessarily since the mapping ϕ is biholomorphic.

Then, for weight $m \geq 3$, we define a mapping \mathcal{L}_m such that

$$\begin{aligned} \mathcal{L}_m(f_{m-1}, g_m, F_m^*) &\equiv \Im g_m(z, u + iF_2(z, \bar{z}, u)) \\ &- 2\Re \langle f_1(z, 0), f_{m-1}(z, u + iF_2(z, \bar{z}, u)) \rangle \\ &- F_m^*(f_1(z, 0), \overline{f_1(z, 0)}, \Re g_2(z, u + iF_2(z, \bar{z}, u))) \\ &= \Re g_2(0, 1)F_m(z, \bar{z}, u) + R_m(z, \bar{z}, u). \end{aligned}$$

This completes the proof. \square

Let CP_k , $k \geq 3$, be a subspace of $\mathbb{C}[z, w] \equiv \mathbb{C}[z^1, \dots, z^n, w]$ such that

$$CP_k = \{g \in \mathbb{C}[z, w] : g(\mu z, \mu^2 w) = \mu^k g(z, w)\}$$

and RP_k , $k \geq 3$, be a subspace of $\mathbb{R}[\Re z, \Im z, \Re w]$ such that

$$RP_k = \{F \in \mathbb{R}[\Re z, \Im z, \Re w] : F(\mu z, \mu \bar{z}, \mu^2 u) = \mu^k F(z, \bar{z}, u)\}.$$

We define a subspace NP_k of RP_k , $k \geq 3$, such that

$$\begin{aligned} NP_k &= \left\{ F \in RP_k : F(z, \bar{z}, u) = \sum_{s, t \geq 2} F_{st}(z, \bar{z}, u) \right. \\ &\quad \left. \text{where } \Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0 \right\}. \end{aligned}$$

We shall use the following notations:

$$\begin{cases} O(m) \equiv \sum_{|I|+|J|+2k+2l \geq m} O(z^I \bar{z}^J w^k \bar{w}^l) \\ O(z^s \bar{z}^t) \equiv \sum_{|I|=s, |J|=t} O(z^I \bar{z}^J) \end{cases}$$

where $k, l, s, t \in \mathbb{N}$, and I, J are multi-indices in \mathbb{N}^n .

Lemma 1.2. *Let $\mathcal{L}_k : HP_{k-1}^n \times HP_k \times NP_k \longrightarrow RP_k$ for $k \geq 3$ be the mapping defined in Lemma 1.1 as follows*

$$\mathcal{L}_k(f, g, F^*) \equiv [\Re\{2\langle f(z, w), Cz \rangle + ig(z, w)\} - F^*(Cz, \overline{Cz}, \rho\Re\chi(z, w))]_{w=u+iF_2(z, \bar{z}, u)}$$

where $HP_k^n = HP_k \times \cdots \times HP_k$ (n times) and

$$\begin{cases} (f, g, F^*) \in HP_{k-1}^n \times HP_k \times NP_k \\ Cz = f_1(z, 0), \quad \rho = \Re g_2(0, 1) \\ \chi(z, w) = \{1 - iF_2(0, 0, 1)\}\{w - iF_2(z, 0, 0)\}. \end{cases}.$$

Then \mathcal{L}_k is surjective for $k \geq 3$ and

$$\begin{aligned} \dim_{\mathbb{R}} \ker \mathcal{L}_3 &= 2n \\ \dim_{\mathbb{R}} \ker \mathcal{L}_4 &= 1 \\ \dim_{\mathbb{R}} \ker \mathcal{L}_k &= 0 \quad \text{for } k \geq 5. \end{aligned}$$

Proof. Note that

$$\varprojlim_k \mathbb{R}[\Re z, \Im z, u]/O(k) \simeq \mathbb{R}[[\Re z, \Im z, u]]$$

where $\varprojlim_k \mathbb{R}[\Re z, \Im z, u]/O(k)$ is the inverse limit of $\mathbb{R}[\Re z, \Im z, u]/O(k)$, $k \in \mathbb{N}$. Hence we can take arbitrary $F_k(z, \bar{z}, u)$ in Lemma 1.1 so that the mapping \mathcal{L}_k , $k \geq 3$, is surjective. Then by Theorem 0.2 the kernel of the mapping \mathcal{L}_k , $k \geq 3$, is parametrized by a, r in the value (C, a, ρ, r) so that

$$\begin{aligned} \dim_{\mathbb{R}} \ker \mathcal{L}_3 &= 2n \\ \dim_{\mathbb{R}} \ker \mathcal{L}_4 &= 1 \\ \dim_{\mathbb{R}} \ker \mathcal{L}_k &= 0 \quad \text{for } k \geq 5. \end{aligned}$$

This completes the proof. \square

The polynomial identities in Lemma 1.1 and Lemma 1.2 yields the following existence and uniqueness theorem.

Theorem 1.3. *Let M be a nondegenerate analytic real hypersurface defined by*

$$v = \sum_{k=2}^{\infty} F_k(z, \bar{z}, u).$$

Then there is a natural isomorphism for each $k \geq 4$ via the mapping \mathcal{L}_k , $k \geq 3$, such that

$$\begin{aligned} &\{F_l \in RP_l : l \leq k\} \times H \\ &\simeq \{(f_{l-1}, g_l) \in CP_{l-1}^n \times CP_l : l \leq k\} \oplus \{F_l^* \in NP_l : l \leq k\}. \end{aligned}$$

Hence there exist unique mappings for each $k \geq 4$ such that

$$\begin{aligned} \nu : \{F_l : l \leq k\} \times H &\longmapsto \{(f_{l-1}, g_l) : l \leq k\} \\ \kappa : \{F_l : l \leq k\} \times H &\longmapsto \{F_l^* : l \leq k\}. \end{aligned}$$

Further, the formal biholomorphic mapping

$$\phi = \left(\sum_{k=1}^{\infty} f_k, \sum_{k=2}^{\infty} g_k \right)$$

converges so that $\phi(M)$ is an analytic real hypersurface defined by the equation

$$v = \langle z, z \rangle + \sum_{k=4}^{\infty} F_k^*(z, \bar{z}, u).$$

Hence, for the special case of $k = \infty$ in Theorem 1.3, we obtain

Theorem 1.4. *Let M be a nondegenerate analytic real hypersurface defined by*

$$v = \sum_{k=2}^{\infty} F_2(z, \bar{z}, u)$$

and $\phi = (\sum_k f_k, \sum_k g_k)$ be a normalization of M such that the real hypersurface $\phi(M)$ is defined in normal form by the equation

$$v = \langle z, z \rangle + \sum_k F_k^*(z, \bar{z}, u).$$

Then the functions $f_{k-1}, g_k, F_k^, k \geq 3$, are given as a finite linear combination of finite multiples of the following factors:*

- (1) *the coefficients of the functions $F_l, l \leq k$,*
- (2) *the constants $C, C^{-1}, \rho, \rho^{-1}, a, r$,*

where (C, a, ρ, r) are the initial value of the normalization ϕ .

II. We shall examine concrete versions of Theorem 1.3 as existence and uniqueness theorem. Let $h = (f, g)$ and $\phi = (\tilde{f}, \tilde{g})$ be holomorphic mappings in $\mathbb{C}^n \times \mathbb{C}$ such that

$$f = \tilde{f} + O(k) \text{ and } g = \tilde{g} + O(k+1).$$

Then we shall write

$$h = \phi + O_{\times}(k+1).$$

Theorem 1.5. *Let M be an analytic real hypersurface with nondegenerate Levi form defined by*

$$(1.3) \quad v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0.$$

If $h = (f, g)$ is a biholomorphic mapping such that

$$\begin{aligned} f(z, w) &= C(z - aw) + f^*(z, w), \\ g(z, w) &= \rho(w + rw^2) + g^*(z, w), \end{aligned}$$

where the functions $f^(z, w)$ and $g^*(z, w)$ satisfy the condition:*

$$(1.4) \quad f^*|_0 = df^*|_0 = g^*|_0 = dg^*|_0 = \Re(g_{w\bar{w}}^*|_0) = 0,$$

and if the transformed real hypersurface $h(M)$ is defined by

$$(1.5) \quad v = F^*(z, \bar{z}, u) + O(k+1),$$

where $v = F^(z, \bar{z}, u)$ is in normal form, then there is a normalization of M , ϕ_{σ} , with initial value $\sigma = (C, a, \rho, r) \in H$ such that*

$$h = \phi_{\sigma} + O_{\times}(k+1).$$

Proof. Note that

$$F_m^* \in NP_m \quad \text{for } 3 \leq m \leq k.$$

Then we obtain, for $3 \leq m \leq k$,

$$\mathcal{L}_m(f_{m-1}, g_m, F_m^*) = \rho F_m(z, \bar{z}, u) + R_m(z, \bar{z}, u)$$

where $R_m(z, \bar{z}, u)$ is a linear combination of multiples of f_{s-1}, g_s, F_s, F_s^* , $s \leq m-1$, and their derivatives. The condition (1.4) makes the mappings \mathcal{L}_k , $k \geq 3$, be one-to-one. Hence the coefficients of the functions

$$f_{m-1}, \quad g_m, \quad F_m^* \quad \text{for } m \leq k$$

are completely determined by the coefficients of the functions F_m , $m \leq k$, and the initial value $\sigma = (C, a, \rho, r)$.

By Lemma 1.1 and Lemma 1.2, the normalization ϕ_σ and the defining function $\phi_\sigma(M)$ is uniquely determined by the value $\sigma = (C, a, \rho, r)$ via the following equalities:

$$\mathcal{L}_m(f_{m-1}^*, g_m^*, F_m^{**}) = \rho F_m(z, \bar{z}, u) + R_m(z, \bar{z}, u)$$

where

$$\phi_\sigma(M) : v = \langle z, z \rangle + F^{**}(z, \bar{z}, u)$$

and

$$\begin{aligned} \phi_\sigma &= (f^*, g^*) \\ f^*(z, w) &= \sum_{m=1}^{\infty} f_m^*(z, w) \\ g^*(z, w) &= \sum_{m=1}^{\infty} g_m^*(z, w) \\ F^{**}(z, \bar{z}, u) &= \sum_{m=4}^{\infty} F_m^{**}(z, \bar{z}, u). \end{aligned}$$

Since ϕ_σ and h have the same initial value, we obtain

$$f_{m-1} = f_{m-1}^*, \quad g_m = g_m^* \quad \text{for } m \leq k$$

so that

$$h = \phi_\sigma + O_\times(k+1).$$

This completes the proof. \square

Lemma 1.6. *Let $\varphi = (f, g)$, $\varphi_1 = (f_1, g_1)$, $\varphi_2 = (f_2, g_2)$ be biholomorphic mappings in $\mathbb{C}^n \times \mathbb{C}$ such that*

$$\begin{cases} f|_0 = g|_0 = \frac{\partial g}{\partial z}|_0 = 0 \\ f_1|_0 = g_1|_0 = \frac{\partial g_1}{\partial z}|_0 = 0 \\ f_2|_0 = g_2|_0 = \frac{\partial g_2}{\partial z}|_0 = 0. \end{cases}$$

Then for each $k \geq 3$

(1) $\varphi^{-1} = \phi^{-1} + O_\times(k)$ whenever

$$\varphi = \phi + O_\times(k),$$

(2) $\varphi_1 \circ \varphi_2 = \phi_1 \circ \phi_2 + O_{\times}(k)$ whenever

$$\begin{aligned}\varphi_1 &= \phi_1 + O_{\times}(k) \\ \varphi_2 &= \phi_2 + O_{\times}(k).\end{aligned}$$

Proof. Let $\varphi = (f, g)$ and $\varphi^{-1} = (f^*, g^*)$ so that we obtain following identities

$$(1.6) \quad \begin{cases} f(f^*(z, w), g^*(z, w)) = z \\ g(f^*(z, w), g^*(z, w)) = w \end{cases}.$$

We expand the identity (1.6) and collect terms of the same weight with the weight decompositions

$$\begin{cases} f(z, w) = \sum_{k=1}^{\infty} f_k(z, w), & g(z, w) = \sum_{k=2}^{\infty} g_k(z, w) \\ f^*(z, w) = \sum_{k=1}^{\infty} f_k^*(z, w), & g^*(z, w) = \sum_{k=2}^{\infty} g_k^*(z, w) \end{cases}.$$

We set

$$C = \left(\frac{\partial f}{\partial z} \Big|_0 \right) \quad \text{and} \quad \rho = \left(\frac{\partial g}{\partial w} \Big|_0 \right).$$

Then we easily see that the function

$$Cf_{m-1}^*(z, w)$$

is given by a finite linear combination of finite multiples of the functions

$$f_1, \dots, f_{m-1}, f_1^*, \dots, f_{m-2}^*, g_2^*, \dots, g_{m-1}^* \quad \text{and their derivatives,}$$

and the function

$$\rho g_m^*(z, w)$$

is given by a finite linear combination of finite multiples of the functions

$$g_2, \dots, g_m, f_1^*, \dots, f_{\lfloor \frac{m}{2} \rfloor}^*, g_2^*, \dots, g_{m-1}^* \quad \text{and their derivatives.}$$

Then, by a simple induction argument, the functions

$$(f_{m-1}^*(z, w), g_m^*(z, w))$$

are given by a finite linear combination of finite multiples of the functions

$$f_1, \dots, f_{m-1}, g_2, \dots, g_m \quad \text{and their derivatives}$$

and

$$C, \quad C^{-1}, \quad \rho, \quad \rho^{-1}.$$

Hence we have proved

$$(1.7) \quad \varphi^{-1} = \phi^{-1} + O_{\times}(k)$$

whenever

$$\varphi = \phi + O_{\times}(k).$$

Since the mapping $\varphi_1 = (f_1, g_1)$ satisfies the condition

$$f_1|_0 = g_1|_0 = \frac{\partial g_1}{\partial z} \Big|_0 = 0,$$

we obtain

$$\varphi_1 \circ \varphi_2 = \varphi_1 \circ \phi_2 + O_{\times}(k)$$

whenever

$$\varphi_2 = \phi_2 + O_{\times}(k).$$

Then, by the result (1.7), we have

$$(1.8) \quad \varphi_2^{-1} \circ \varphi_1^{-1} = \phi_2^{-1} \circ \varphi_1^{-1} + O_{\times}(k).$$

Note that the mapping $\phi_2^{-1} = (f_2^*, g_2^*)$ satisfies the condition

$$(1.9) \quad f_2^*|_0 = g_2^*|_0 = \left. \frac{\partial g_2^*}{\partial z} \right|_0 = 0,$$

whenever $\phi_2 = (f_2, g_2)$ satisfies

$$f_2|_0 = g_2|_0 = \left. \frac{\partial g_2}{\partial z} \right|_0 = 0.$$

By the condition (1.9), we obtain

$$(1.10) \quad \phi_2^{-1} \circ \varphi_1^{-1} = \phi_2^{-1} \circ \phi_1^{-1} + O_{\times}(k)$$

whenever

$$\varphi_1^{-1} = \phi_1^{-1} + O_{\times}(k).$$

By the result (1.7), the equalities (1.8) and (1.10) yields

$$\begin{aligned} \varphi_2^{-1} \circ \varphi_1^{-1} &= \phi_2^{-1} \circ \varphi_1^{-1} + O_{\times}(k) \\ &= \phi_2^{-1} \circ \phi_1^{-1} + O_{\times}(k). \end{aligned}$$

Once more by the result (1.7), we obtain

$$\varphi_1 \circ \varphi_2 = \phi_1 \circ \phi_2 + O_{\times}(k).$$

This completes the proof. \square

Lemma 1.7. *Let M be a real hypersurface defined by*

$$\begin{aligned} v &= F(z, \bar{z}, u), \\ F|_0 &= F_z|_0 = F_{\bar{z}}|_0 = 0. \end{aligned}$$

Let $\phi_1 = (f, g), \phi_2 = (f', g')$ be biholomorphic mappings of M such that

$$\begin{aligned} f|_0 &= g|_0 = g_z|_0 = 0 \\ f'|_0 &= g'|_0 = g'_z|_0 = 0 \end{aligned}$$

and, for $k \geq 3$,

$$\phi_1 = \phi_2 + O_{\times}(k+1).$$

Suppose that the transformed real hypersurfaces $\phi_1(M), \phi_2(M)$ are defined by

$$v = G(z, \bar{z}, u), \quad v = G'(z, \bar{z}, u).$$

Then

$$G(z, \bar{z}, u) = G'(z, \bar{z}, u) + O(k+1).$$

Proof. Note that

$$g(z, w) = g'(z, w) + O(k+1)$$

and

$$F(z, \bar{z}, u) = O(2)$$

which yields

$$g(z, u + iF(z, \bar{z}, u)) = g'(z, u + iF(z, \bar{z}, u)) + O(k+1).$$

Then we obtain

$$\begin{aligned} G(z^*, \bar{z}^*, u^*) &= \frac{1}{2i} \{g(z, w) - \bar{g}(\bar{z}, \bar{w})\}|_{(z,w)=\phi_1^{-1}(z^*, w^*)} \\ &= \frac{1}{2i} \{g'(z, w) - \bar{g}'(\bar{z}, \bar{w})\}|_{(z,w)=\phi_1^{-1}(z^*, w^*)} + O(k+1) \end{aligned}$$

By Lemma 1.6, we have

$$\phi_1^{-1} = \phi_2^{-1} + O_{\times}(k+1)$$

and note that

$$g_z|_0 = g'_z|_0 = 0.$$

Then we obtain

$$\begin{aligned} G(z^*, \bar{z}^*, u^*) &= \frac{1}{2i} \{g'(z, w) - \bar{g}'(\bar{z}, \bar{w})\}|_{(z,w)=\phi_1^{-1}(z^*, w^*)} + O(k+1) \\ &= \frac{1}{2i} \{g'(z, w) - \bar{g}'(\bar{z}, \bar{w})\}|_{(z,w)=\phi_2^{-1}(z^*, w^*)} + O(k+1). \end{aligned}$$

Note that

$$G'(z^*, \bar{z}^*, u^*) = \frac{1}{2i} \{g'(z, w) - \bar{g}'(\bar{z}, \bar{w})\}|_{(z,w)=\phi_2^{-1}(z^*, w^*)}.$$

Thus we obtain

$$G(z^*, \bar{z}^*, u^*) = G'(z^*, \bar{z}^*, u^*) + O(k+1).$$

This completes the proof. \square

Hence we obtain the following theorem

Theorem 1.8. *Let M be an analytic real hypersurface in Theorem 1.5 and $h = (f, g)$ be a biholomorphic mapping such that*

$$h = \phi + O_{\times}(k+1)$$

where ϕ is a normalization of M with initial value $\sigma = (C, a, \rho, r)$. Suppose that

$$\begin{aligned} h(M) : v &= G(z, \bar{z}, u) \\ \phi(M) : v &= G'(z, \bar{z}, u) \end{aligned}$$

Then

$$G(z, \bar{z}, u) = G'(z, \bar{z}, u) + O(k+1).$$

III. Let ϕ be a fractional linear mapping such that

$$(1.11) \quad \phi = \phi_\sigma : \begin{cases} z^* = \frac{C(z-aw)}{1+2i\langle z,a \rangle - w(r+i\langle a,a \rangle)} \\ w^* = \frac{\rho w}{1+2i\langle z,a \rangle - w(r+i\langle a,a \rangle)} \end{cases}$$

where the constants $\sigma = (C, a, \rho, r)$ satisfy

$$\begin{aligned} a &\in \mathbb{C}^n, \quad \rho \neq 0, \quad \rho, r \in \mathbb{R}, \\ C &\in GL(n; \mathbb{C}), \quad \langle Cz, Cz \rangle = \rho \langle z, z \rangle. \end{aligned}$$

Note that the mapping ϕ decomposes to

$$\phi = \varphi \circ \psi,$$

where

$$(1.12) \quad \psi : \begin{cases} z^* = \frac{z-aw}{1+2i\langle z,a \rangle - i\langle a,a \rangle w} \\ w^* = \frac{w}{1+2i\langle z,a \rangle - i\langle a,a \rangle w} \end{cases} \quad \text{and} \quad \varphi : \begin{cases} z^* = \frac{Cz}{1-rw} \\ w^* = \frac{\rho w}{1-rw} \end{cases}.$$

We easily verify

$$\phi^*(v - \langle z, z \rangle) = (v - \langle z, z \rangle) \rho (1 + \delta)^{-1} (1 + \bar{\delta})^{-1},$$

where

$$1 + \delta = 1 + 2i\langle z, a \rangle - (r + i\langle a, a \rangle)w.$$

By Theorem 0.2, each element of the isotropy subgroup of the automorphism group of a real hyperquadric $v = \langle z, z \rangle$ is given by a fractional linear mapping ϕ_σ in (1.11).

Theorem 1.9. *Let M be an analytic real hypersurface in normal form such that*

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + \sum_{k \geq l+1} F_k(z, \bar{z}, u),$$

where

$$F_l(z, \bar{z}, u) \neq 0.$$

Let N_σ be a normalization of M and ϕ_σ be an automorphism of the real hyperquadric with the initial value $\sigma = (C, a, \rho, r) \in H$. Suppose that the transformed real hypersurface $N_\sigma(M)$ is defined by

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u).$$

Then

- (1) $N_\sigma = \phi_\sigma + O_\times(l+1)$,
- (2) $F^*(z, \bar{z}, u) = \rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) + O(l+1)$.

Proof. We easily compute ψ^{-1} and φ^{-1} as follows:

$$\psi^{-1} : \begin{cases} z = \frac{z^* + aw^*}{1-2i\langle z^*, a \rangle - i\langle a, a \rangle w^*} \\ w = \frac{w^*}{1-2i\langle z^*, a \rangle - i\langle a, a \rangle w^*} \end{cases} \quad \text{and} \quad \varphi^{-1} : \begin{cases} z = \frac{C^{-1}z^*}{1+r\rho^{-1}w^*} \\ w = \frac{\rho^{-1}w^*}{1+r\rho^{-1}w^*} \end{cases}.$$

Since $\phi_\sigma^{-1} = \psi^{-1} \circ \varphi^{-1}$, we obtain

$$\phi_\sigma^{-1} = \phi_{\sigma^{-1}} : \begin{cases} z = \frac{C^{-1}(z^* + \rho^{-1}Caw^*)}{1-2i\langle z^*, \rho^{-1}Ca \rangle - w^*(-r\rho^{-1} + i\langle \rho^{-1}Ca, \rho^{-1}Ca \rangle)} \\ w = \frac{\rho^{-1}w^*}{1-2i\langle z^*, \rho^{-1}Ca \rangle - w^*(-r\rho^{-1} + i\langle \rho^{-1}Ca, \rho^{-1}Ca \rangle)} \end{cases}$$

where

$$\sigma^{-1} = (C^{-1}, -\rho^{-1}Ca, \rho^{-1}, -r\rho^{-1}) \in H.$$

Thus we have

$$v - \langle z, z \rangle = (v^* - \langle z^*, z^* \rangle) \rho^{-1} (1 - \delta^*)^{-1} (1 - \overline{\delta^*})^{-1},$$

where

$$1 - \delta^* = 1 - 2i\rho^{-1} \langle z^*, Ca \rangle - \rho^{-1} w^* (-r + i\langle a, a \rangle).$$

By the mapping ϕ_σ in the decomposition $N_\sigma = E \circ \phi_\sigma$, we obtain that

$$\begin{aligned} v^* &= \langle z^*, z^* \rangle + \rho F_l(C^{-1}z^*, \overline{C^{-1}z^*}, \rho^{-1}u^*) + \sum_{|I|+|J|+2k \geq l+1} O(z^{*I} \bar{z}^{*J} u^{*k}), \\ &= \langle z^*, z^* \rangle + \rho F_l(C^{-1}z^*, \overline{C^{-1}z^*}, \rho^{-1}u^*) + O(l+1). \end{aligned}$$

Note that the following real hypersurface is in normal form:

$$v = \langle z, z \rangle + \rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u).$$

By Theorem 1.5, N_σ agrees with ϕ_σ up to weight l so that

$$N_\sigma = \phi_\sigma + O_\times(l+1).$$

Then Theorem 1.8 implies

$$F^*(z, \bar{z}, u) = \rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) + O(l+1).$$

This completes the proof. \square

Since $\det(C) \neq 0$ and $\rho \neq 0$, the weight l of the real hypersurface M in Theorem 1.9 is the non-vanishing lowest weight of the function $F^*(z, \bar{z}, u)$ regardless of the initial value σ , where

$$N_\sigma(M) : v = \langle z, z \rangle + F^*(z, \bar{z}, u).$$

Thus the non-vanishing lowest weight l is an invariant of M under germs of biholomorphic mappings.

Corollary 1.10. *Let M be an analytic real hypersurface in normal form which is invariant under the action of all normalizations. Then necessarily, M is a real hyperquadric.*

Proof. Suppose that M is not a real hyperquadric defined by

$$v = \langle z, z \rangle + F(z, \bar{z}, u).$$

Then there is a positive integer l such that

$$F(z, \bar{z}, u) = F_l(z, \bar{z}, u) + O(l+1),$$

where

$$F_l(z, \bar{z}, u) \neq 0.$$

We take an initial value $\sigma = (C, a, \rho, r)$ such that

$$C = \sqrt{\rho}, \quad a = 0, \quad \rho > 0, \quad r = 0.$$

Let ϕ be a normalization of M with initial value $\sigma = (C, a, \rho, r)$ such that

$$\phi(M) : v = \langle z, z \rangle + F^*(z, \bar{z}, u)$$

Then by Theorem 1.9, we obtain

$$\begin{aligned} F^*(z, \bar{z}, u) &= \rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) + O(l+1) \\ &= \rho^{\frac{2-l}{2}} F_l(z, \bar{z}, u) + O(l+1), \end{aligned}$$

where

$$\rho > 0, \quad \sigma = (\sqrt{\rho}, 0, \rho, 0) \in H.$$

Since $l \geq 4$ necessarily, the assumption implies, for all $\sigma \in H$,

$$F^*(z, \bar{z}, u) = F_l(z, \bar{z}, u) + O(l+1).$$

Hence we obtain

$$F_l(z, \bar{z}, u) = 0.$$

This is a contradiction to the choice of the integer l . Thus we have showed $F(z, \bar{z}, u) = 0$ so that M is a real hyperquadric. \square

Lemma 1.11 (Kruzhilin). *Let M be an analytic real hypersurface in normal form. Then each normalization N_σ with initial value $\sigma \in H$ satisfies the following relation:*

$$N_\sigma = \phi_\sigma + O_\times(5)$$

where ϕ_σ is a local automorphism of a real hyperquadric with initial value $\sigma \in H$.

Proof. Note that M is of the form:

$$v = \langle z, z \rangle + F_{22}(z, \bar{z}, 0) + O(5),$$

where $F_{22}(z, \bar{z}, 0)$ is of weight 4. By Theorem 1.9, the normalization N_σ agrees with the mapping ϕ_σ in low order terms such that

$$N_\sigma = \phi_\sigma + O_\times(5).$$

This completes the proof. \square

Theorem 1.12. *Let M be an analytic real hypersurface in normal form and N_{σ_2} be a normalization of M with initial value $\sigma_2 \in H$. Let N_{σ_1} be a normalization of $N_{\sigma_2}(M)$ with initial value $\sigma_1 \in H$ and $N_{\sigma_1\sigma_2}$ be a normalization of M with initial value $\sigma_1\sigma_2 \in H$. Then*

$$N_{\sigma_1} \circ N_{\sigma_2} = N_{\sigma_1\sigma_2}.$$

Proof. Since M is in normal form, Lemma 1.11 yields

$$N_\sigma = \phi_\sigma + O_\times(5)$$

for all $\sigma \in H$. Hence we obtain

$$\begin{aligned} N_{\sigma_1} \circ N_{\sigma_2} &= \phi_{\sigma_1} \circ \phi_{\sigma_2} + O_\times(5) \\ &= \phi_{\sigma_1\sigma_2} + O_\times(5) \\ &= N_{\sigma_1\sigma_2} + O_\times(5). \end{aligned}$$

Note that the initial value σ of a normalization $N_\sigma = (f, g)$ is completely determined by the terms

$$\begin{array}{lll} f_1(z, w), & f_2(z, w), & f_3(z, w) \\ g_2(z, w), & g_3(z, w), & g_4(z, w) \end{array}$$

where

$$f(z, w) = \sum_{k=1}^{\infty} f_k(z, w), \quad g(z, w) = \sum_{k=2}^{\infty} g_k(z, w).$$

Then Theorem 0.2 yields

$$N_{\sigma_1} \circ N_{\sigma_2} = N_{\sigma_1 \sigma_2}.$$

This completes the proof. \square

IV. For a real hypersurface M in normal form, we define the isotropy subgroup $H(M)$ of M as follows:

$$H(M) = \{\sigma \in H : N_\sigma(M) = M\}.$$

It is known that $H(M)$ is a Lie group(cf. [Pa2]).

Lemma 1.13. *Let M be a nondegenerate analytic real hypersurface defined by*

$$v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0$$

and M' be another analytic real hypersurface in normal form which is biholomorphic to M near the origin. Then there is an element $\sigma \in H$ and a normalization N_σ such that $M' = N_\sigma(M)$. Suppose in addition that M is in normal form. Then

$$M' = N_{\sigma'}(M) \quad \text{if and only if} \quad \sigma' \in H(M')\sigma = \sigma H(M),$$

where $H(M')\sigma$ is a left coset of $H(M')$ in H and $\sigma H(M)$ is a right coset of $H(M)$ in H .

Proof. Since M, M' are biholomorphic, we take a biholomorphic mapping $N = (f, g)$. Then by Theorem 0.2 we have $N = N_\sigma$ with $\sigma = (C, a, \rho, r)$, where

$$C = \frac{\partial f}{\partial z}\bigg|_0, \quad -Ca = \frac{\partial f}{\partial w}\bigg|_0, \quad \rho = \left(\frac{\partial g}{\partial w}\bigg|_0\right), \quad 2\rho r = \Re\left(\frac{\partial^2 g}{\partial w^2}\bigg|_0\right).$$

Suppose that M is in normal form. Then by Theorem 1.12 we have

$$\begin{aligned} M' &= N_{\sigma'}(M) = N_{\sigma' \sigma^{-1}} \circ N_\sigma(M) \\ &= N_{\sigma' \sigma^{-1}}(M'), \end{aligned}$$

which yields $\sigma' \sigma^{-1} \in H(M')$. Hence we obtain

$$\sigma' \in H(M')\sigma.$$

Clearly, the converse is also true.

Suppose that M, M' are both in normal form. Then $M' = N_\sigma(M)$ if and only if $M = N_{\sigma^{-1}}(M')$. Thus we obtain

$$\sigma^{-1} H(M') \sigma \subset H(M), \quad \sigma H(M) \sigma^{-1} \subset H(M')$$

so that

$$\sigma H(M) = H(M')\sigma$$

This completes the proof. \square

Theorem 1.14. *Let \mathfrak{N} be the set of real hypersurfaces in normal form with signature $(s, n-s)$, $\frac{n}{2} \leq s \leq n$. Then \mathfrak{N} is an H -space under the group H action via normalizations.*

Proof. Let M be a real hypersurface in normal form. Then, by Theorem 0.2, the normalization of M with identity initial value is necessarily the identity map. Further, by Theorem 1.12, we have

$$N_{\sigma_1} \circ N_{\sigma_2}(M) = N_{\sigma_1 \sigma_2}(M).$$

Thus the set \mathfrak{N} is an H -space under the group H action via normalizations. This completes the proof. \square

Let \mathfrak{N}/H be the orbit space of the H -space \mathfrak{N} . Suppose that M, M'' be analytic real hypersurfaces in normal form which are biholomorphic (even formally) to each other near the origin. Then, by Lemma 1.13, there is a normalization N_σ with an initial value $\sigma \in H$ such that

$$M'' = N_\sigma(M).$$

Hence the orbit space \mathfrak{N}/H may be isomorphic to the orbit space \mathfrak{M} of germs of strictly pseudoconvex analytic real hypersurfaces under germs of biholomorphic mappings.

We have already seen several local invariants under germs of biholomorphic mappings such as the signature $(s, n - s)$ of Levi form on M and the nonvanishing lowest weight l of the defining function of M in a normal coordinate. It is known that every CR diffeomorphism of class C^1 between analytic real hypersurfaces with nondegenerate Levi form is necessarily real-analytic so as to extend to a biholomorphic mapping on a neighborhood (cf. [Pi], [Le], [BJT]). Therefore, we may conclude that the orbit space \mathfrak{N}/H is a classifying space of germs of nondegenerate analytic real hypersurfaces. Hence any function defined via Chern-Moser normal coordinate is a local CR invariant whenever it does not depend on the choice of normal coordinates. We refer to Wells [Wl] for a history of many aspects of Cauchy-Riemann invariants.

2. UMBILIC POINTS ON ANALYTIC REAL HYPERSURFACES

I. Let M be an analytic real hypersurface and p be a point on M such that M is, in normal coordinate with center at p , defined by

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u),$$

where

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0.$$

Note that $F_{22} = F_{23} = F_{33} = 0$ if $\dim M = 3$. By Theorem 1.9, the following definition makes sense:

- (1) If $\dim M = 3$, the point $p \in M$ is called umbilic whenever

$$F_{42}(z, \bar{z}, 0) = F_{24}(z, \bar{z}, 0) = 0.$$

- (2) If $\dim M \geq 5$, the point $p \in M$ is called umbilic whenever

$$F_{22}(z, \bar{z}, 0) = 0.$$

Let N_σ be a normalization of M and $\phi_\sigma = \varphi \circ \psi$ be an automorphism of a real hyperquadric. We have a decomposition of N_σ (cf. [Pa2]):

$$N_\sigma = \phi \circ E \circ \psi$$

where E is a normalization of $\psi(M)$ with identity initial value. Then we easily verify for each $k \geq 3$

$$N_\sigma = \phi_\sigma + O_\times(k+1) \quad \text{if and only if} \quad E = id + O_\times(k+1).$$

Theorem 2.1. *Let M be an analytic real hypersurface of dimension 3 in normal form. Let N_σ be a normalization of M such that $a \neq 0$ in $\sigma = (C, a, \rho, r)$. Then the nonumbilicity of the origin $0 \in M$ is equivalent to the following condition:*

$$N_\sigma = \phi_\sigma + O_\times(7) \quad \text{and} \quad N_\sigma \neq \phi_\sigma + O_\times(8).$$

Proof. It may suffice to show that the normalization E in $N_\sigma = \varphi \circ E \circ \psi$ satisfies

$$(2.1) \quad E = id + O_\times(7) \quad \text{and} \quad E \neq id + O_\times(8)$$

if and only if the origin is nonumbilic.

Suppose that M is defined by

$$v = z\bar{z} + bz^4\bar{z}^2 + \bar{b}z^2\bar{z}^4 + cz^5\bar{z}^2 + \bar{c}z^2\bar{z}^5 + dz^4\bar{z}^3 + \bar{d}z^3\bar{z}^4 + O(8),$$

where

$$b, c, d \in \mathbb{C}.$$

By the mapping ψ , M is transformed up to weight 7 to a real hypersurface as follows:

$$\begin{aligned} v = & z\bar{z} + bz^4\bar{z}^2 + \bar{b}z^2\bar{z}^4 + (c + 4ib\bar{a})z^5\bar{z}^2 + (\overline{c + 4ib\bar{a}})z^2\bar{z}^5 \\ & + (d + 2iba)z^4\bar{z}^3 + (\overline{d + 2iba})z^3\bar{z}^4 \\ & + 4bauz^3\bar{z}^2 + 4\bar{b}\bar{a}uz^2\bar{z}^3 + 2b\bar{a}uz^4\bar{z} + 2\bar{b}auz\bar{z}^4 + O(8) \end{aligned}$$

By Theorem 1.5, we obtain the normalization E up to weight 7 as follows:

$$\begin{aligned} z^* &= z + 2\bar{a}bz^4w - 2iabz^2w^2 - \frac{1}{3i}\bar{a}\bar{b}w^3 + O(7) \\ w^* &= w + \frac{2}{3}abzw^3 + O(8), \end{aligned}$$

and M is transformed up to weight 7 to a real hypersurface as follows:

$$\begin{aligned} v = & z\bar{z} + bz^4\bar{z}^2 + \bar{b}z^2\bar{z}^4 + (c + 2ib\bar{a})z^5\bar{z}^2 + (\overline{c + 2ib\bar{a}})z^2\bar{z}^5 \\ (2.2) \quad & + (d + \frac{2}{3}iba)z^4\bar{z}^3 + (\overline{d + \frac{2}{3}iba})z^3\bar{z}^4 + O(8) \end{aligned}$$

Since $a \neq 0$, the normalization E satisfies the condition (2.1) if and only if $b \neq 0$. This completes the proof. \square

Corollary 2.2 (Moser). *Let M be a real hypersurface of dimension 3 with a non-umbilic point $p \in M$. Then there is a normal coordinate with center at $p \in M$ such that*

$$v = z\bar{z} + F_{42}(z, \bar{z}, u) + F_{24}(z, \bar{z}, u) + \sum_{\min(s,t) \geq 2, s+t \geq 7} F_{st}(z, \bar{z}, u)$$

where

$$(2.3) \quad F_{24}(z, \bar{z}, 0) = z^2\bar{z}^4, \quad \Re \left\{ z^2 \left(\frac{\partial F_{24}}{\partial u} \right) (z, \bar{z}, 0) \right\} = 0, \quad F_{43}(z, \bar{z}, 0) = 0.$$

Further, if two real hypersurfaces are of the reduced normal form and they are biholomorphic near the origin, then they are related by a mapping as follows:

$$z^* = \pm z, \quad w^* = w.$$

The same is true when we replace the condition (2.3) by the following condition:

$$F_{24}(z, \bar{z}, 0) = z^2 \bar{z}^4, \quad \Re \left\{ z^2 \left(\frac{\partial F_{24}}{\partial u} \right) (z, \bar{z}, 0) \right\} = 0, \quad F_{52}(z, \bar{z}, 0) = 0.$$

Proof. Since the point p is nonumbilic, we take a normalization N_σ (cf. Theorem 2.1) with initial value

$$\sigma = (1, a, 1, 0) \\ a = \frac{3id}{2b} \quad \left(\text{resp.} \quad a = -\frac{i\bar{c}}{2\bar{b}} \right)$$

where we assume that

$$\begin{aligned} F_{42}|_{u=0} &= bz^4 \bar{z}^2 \neq 0 \\ F_{52}|_{u=0} &= cz^5 \bar{z}^2 \\ F_{43}|_{u=0} &= dz^4 \bar{z}^3. \end{aligned}$$

Then we obtain

$$(2.4) \quad F_{43}^*|_{u=0} = 0 \quad (\text{resp.} \quad F_{52}^*|_{u=0} = 0).$$

Note that $a = 0$ necessarily if $F_{43}|_{u=0} = 0$ (resp. $F_{52}|_{u=0} = 0$).

Suppose that $F_{43}|_{u=0} = 0$ (resp. $F_{52}|_{u=0} = 0$). Then we take a normalization N_σ with initial value $\sigma = (\alpha, 0, \alpha\bar{\alpha}, 0)$:

$$z^* = \alpha z, \quad w^* = \alpha\bar{\alpha}w,$$

where

$$\alpha = \pm \left(\frac{b^6}{\bar{b}^2} \right)^{1/4}.$$

Then we obtain

$$F_{24}^*|_{u=0} = z^2 \bar{z}^4.$$

Note that $a = 0$ and $\alpha = \pm 1$ necessarily if

$$\begin{aligned} &F_{43}|_{u=0} = 0 \text{ and } F_{24}|_{u=0} = z^2 \bar{z}^4 \\ &(\text{resp.} \quad F_{52}|_{u=0} = 0 \text{ and } F_{24}|_{u=0} = z^2 \bar{z}^4). \end{aligned}$$

Then we carry out another normalization N_σ with initial value $\sigma = (1, 0, 1, r)$:

$$z^* = \frac{z}{1-rw}, \quad w^* = \frac{w}{1-rw},$$

on a real hypersurface M defined up to weight 8 by the following equation:

$$v = z\bar{z} + z^4 \bar{z}^2 + z^2 \bar{z}^4 + duz^4 \bar{z}^2 + \bar{d}uz^2 \bar{z}^4 + O(|z|^7) + O(9).$$

Then M is transformed up to weight 8 to a real hypersurface as follows:

$$v = z\bar{z} + z^4 \bar{z}^2 + z^2 \bar{z}^4 + (d-4r)uz^4 \bar{z}^2 + (\bar{d}-4r)uz^2 \bar{z}^4 + O(|z|^7) + O(9).$$

Taking

$$r = \Re \left(\frac{d}{4} \right)$$

yields

$$\Re \left(\frac{\partial F_{24}^*}{\partial u} \Big|_{u=0} \right) = 0.$$

Thus there is a normal coordinate at a nonumbilic point on M such that

$$\begin{aligned} F_{24}|_{u=0} = z^2 \bar{z}^4, \quad \Re \left(z^2 \frac{\partial F_{24}}{\partial u} \Big|_{u=0} \right) = 0, \quad F_{43}|_{u=0} = 0 \\ \left(\text{resp.} \quad F_{24}|_{u=0} = z^2 \bar{z}^4, \quad \Re \left(z^2 \frac{\partial F_{24}}{\partial u} \Big|_{u=0} \right) = 0, \quad F_{52}|_{u=0} = 0. \right) \end{aligned}$$

Clearly, a normalization N_σ between these normal coordinates has its initial value such as

$$\sigma = (\pm 1, 0, 1, 0).$$

This completes the proof. \square

II. We keep the following conventions unless explicitly specified otherwise.

- (1) All Greek indices run from 1 to n .
- (2) The summation convention over any repeated Greek indices.
- (3) Complex conjugation may be indicated by bars over Greek indices. For instance, $z^{\bar{\alpha}} \equiv \overline{z^\alpha}$, $c_{\bar{\alpha}\bar{\beta}} \equiv \overline{c_{\alpha\beta}}$. We may use \bar{z}^α for $\overline{z^\alpha}$ with a Greek index, but may not with a specific numerical value as in $(\bar{z}^\alpha) = (z^1, \dots, z^n)$. We reserve \bar{z}^k for $(\bar{z})^k$, $k \in \mathbb{N}$, or for a notational abbreviation such as $O(z^s \bar{z}^t)$, $s, t \in \mathbb{N}$.
- (4) We shall raise and lower Greek indices by using the following matrix

$$\text{diag}\{\underbrace{1, \dots, 1}_e, \underbrace{-1, \dots, -1}_{n-e}\}$$

where $(e, n - e)$ is the signature of the quadric $\langle z, z \rangle$. If necessary, we shall indicate the locations of upper indices by dots in order as in $A_{\alpha\beta..}^{\gamma\delta}$.

We shall examine the normalization E in the decomposition $N_\sigma = \varphi \circ E \circ \psi$ in low order terms for the case of $\dim M \geq 5$.

Lemma 2.3. *Let M be an analytic real hypersurface of dimension ≥ 5 in normal form, which is defined up to weight 6 by the following equation:*

$$\begin{aligned} v = \langle z, z \rangle + A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} + u B_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\ + C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^\gamma z^{\bar{\delta}} z^{\bar{\eta}} + C_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} z^{\bar{\eta}} \\ + D_{\alpha\beta\gamma\bar{\delta}\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^{\bar{\delta}} z^{\bar{\eta}} z^{\bar{\xi}} + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + O(6), \end{aligned}$$

where

$$\begin{aligned} \overline{A_{\alpha\beta\bar{\gamma}\bar{\delta}}} = A_{\gamma\delta\bar{\alpha}\bar{\beta}}, \quad \overline{B_{\alpha\beta\bar{\gamma}\bar{\delta}}} = B_{\gamma\delta\bar{\alpha}\bar{\beta}}, \quad \overline{C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}}} = C_{\delta\eta\bar{\alpha}\bar{\beta}\bar{\gamma}}, \quad \overline{D_{\alpha\beta\gamma\bar{\delta}\bar{\eta}\bar{\xi}}} = D_{\delta\eta\bar{\xi}\bar{\alpha}\bar{\beta}\bar{\gamma}}, \\ A_{\alpha\beta.\bar{\delta}}^\alpha = B_{\alpha\beta.\bar{\delta}}^\alpha = 0, \quad C_{\alpha\beta..\bar{\eta}}^{\alpha\beta} = C_{\alpha\beta\gamma..}^{\alpha\beta} = 0, \quad D_{\alpha\beta\gamma...}^{\alpha\beta\gamma} = 0, \end{aligned}$$

and all barred and unbarred indices are respectively symmetric. Let $N_\sigma = \varphi \circ E \circ \psi$ be a normalization with initial value $\sigma = (C, a, \rho, r)$. Then the normalizing mapping

E is given up to weight 6 as follows:

$$\begin{aligned}
z^{*\alpha} &= z^\alpha + 2g^{\alpha\bar{\delta}} A_{\beta\gamma\bar{\delta}\bar{\eta}} a^{\bar{\eta}} z^\beta z^\gamma w + 4ig^{\alpha\bar{\zeta}} A_{\beta\gamma\bar{\eta}\bar{\zeta}} a^{\bar{\eta}} z^\beta z^\gamma \langle z, a \rangle w \\
&\quad + 2g^{\alpha\bar{\eta}} C_{\beta\gamma\bar{\delta}\bar{\eta}\bar{\xi}} z^\beta z^\gamma z^\delta a^{\bar{\xi}} w - 8iz^\alpha A_{\beta\gamma\bar{\eta}\bar{\xi}} z^\beta z^\gamma a^{\bar{\eta}} a^{\bar{\xi}} w \\
&\quad + 2g^{\alpha\bar{\delta}} A_{\beta\gamma\bar{\delta}\bar{\eta}} z^\beta z^\gamma a^{\bar{\eta}} w^2 \\
&\quad - \frac{3i}{n+2} \cdot g^{\alpha\bar{\delta}} \{ C_{\eta\beta\gamma\cdot\bar{\delta}}^\eta z^\beta a^\gamma + C_{\eta\beta\cdot\bar{\delta}\bar{\gamma}}^\eta z^\beta a^{\bar{\gamma}} \} w^2 + O(6) \\
w^* &= w - 4iA_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta a^{\bar{\gamma}} a^{\bar{\delta}} w^2 + O(7).
\end{aligned}$$

Proof. By the mapping ψ in the decomposition of $N_\sigma = \varphi \circ E \circ \psi$, M is transformed up to weight 6 to a real hypersurface as follows:

$$\begin{aligned}
v &= \langle z, z \rangle + F_{02}(z, \bar{z}, u) + F_{20}(z, \bar{z}, u) \\
&\quad + F_{12}(z, \bar{z}, u) + F_{21}(z, \bar{z}, u) + F_{13}(z, \bar{z}, u) + F_{31}(z, \bar{z}, u) \\
&\quad + F_{22}(z, \bar{z}, u) + F_{23}(z, \bar{z}, u) + F_{32}(z, \bar{z}, u) \\
&\quad + F_{33}(z, \bar{z}, u) + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + O(7),
\end{aligned}$$

where

$$\begin{aligned}
F_{02}(z, \bar{z}, u) &= 4u^2 A_{\alpha\beta\bar{\gamma}\bar{\delta}} a^\alpha a^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
F_{12}(z, \bar{z}, u) &= 2u A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
F_{13}(z, \bar{z}, u) &= -4iu \langle a, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^{\bar{\gamma}} z^{\bar{\delta}} + 4iu \langle z, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} a^\alpha a^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
&\quad + 2u C_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha a^\beta z^{\bar{\gamma}} z^{\bar{\delta}} z^{\bar{\eta}} \\
F_{22}(z, \bar{z}, u) &= A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} + 4iu \langle z, a \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
&\quad - 4iu \langle a, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} a^{\bar{\delta}} + u B_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
&\quad + 3u C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} z^\alpha z^\beta a^\gamma z^{\bar{\delta}} z^{\bar{\eta}} + 3u C_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} a^{\bar{\eta}} \\
F_{23}(z, \bar{z}, u) &= C_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} z^{\bar{\eta}} - 2i \langle a, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
&\quad + 2i \langle z, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
F_{33}(z, \bar{z}, u) &= D_{\alpha\beta\gamma\bar{\delta}\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^{\bar{\delta}} z^{\bar{\eta}} z^{\bar{\xi}} - 2 \langle a, a \rangle \langle z, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
&\quad + 4 \langle z, a \rangle \langle a, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} - 4 \langle z, z \rangle \langle z, a \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^{\bar{\gamma}} z^{\bar{\delta}} \\
&\quad - 4 \langle z, z \rangle \langle a, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^{\bar{\gamma}} a^{\bar{\delta}} + 4 \langle z, z \rangle^2 A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^{\bar{\gamma}} a^{\bar{\delta}} \\
&\quad - 2i \langle a, z \rangle C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^\gamma z^{\bar{\delta}} z^{\bar{\eta}} + 2i \langle z, a \rangle C_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} z^{\bar{\eta}} \\
&\quad + 3i \langle z, z \rangle C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} z^\alpha z^\beta a^\gamma z^{\bar{\delta}} z^{\bar{\eta}} - 3i \langle z, z \rangle C_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^{\bar{\gamma}} z^{\bar{\delta}} a^{\bar{\eta}}
\end{aligned}$$

Since M is in normal form, $E = id + O_\times(5)$ by Lemma 1.11. So the function $p(u)$ satisfies

$$p(u) = \frac{1}{2} p''(0) u^2 + O(6).$$

Let $g(z, w)$ be a holomorphic function(cf. Lemma 3 in the paper [Pa2]) defined by

$$\begin{aligned} g(z, w) - g(0, w) &= -2iF(p(w), \overline{p}(w), w) \\ &\quad + 2iF\left(z + p(w), \overline{p}(w), w + \frac{1}{2}\{g(z, w) - g(0, w)\}\right) \\ g(0, w) &= iF(p(w), \overline{p}(w), w). \end{aligned}$$

Then the function $g(0, u)$ satisfies

$$g(0, u) = iF(p(u), \overline{p}(u), u) = O(8).$$

Hence the holomorphic function $g(z, w)$ is implicitly defined up to weight 8 as follows:

$$g = 2iF(z + p(w), \overline{p}(w), w + \frac{1}{2}g) + O(8).$$

Thus we obtain

$$\begin{aligned} g &= 2iF(z + p(w), \overline{p}(w), w + \frac{1}{2}g) + O(8) \\ &= i\langle z, p''(0) \rangle w^2 + 2iF(z, 0, w + \frac{1}{2}g) + O(7) \\ &= i\langle z, p''(0) \rangle w^2 + 4i(w + \frac{1}{2}g)^2 A_{\alpha\beta\overline{\gamma}\overline{\delta}} z^\alpha z^\beta a^\gamma a^{\overline{\delta}} + O(7) \\ &= i\langle z, p''(0) \rangle w^2 + 4iw^2 A_{\alpha\beta\overline{\gamma}\overline{\delta}} z^\alpha z^\beta a^\gamma a^{\overline{\delta}} + O(7). \end{aligned}$$

We carry out the following mapping:

$$\begin{aligned} z &= z^* + \frac{1}{2}p''(0)w^{*2} + O(6) \\ w &= w^* + i\langle z^*, p''(0) \rangle w^{*2} + 4iw^{*2} A_{\alpha\beta\overline{\gamma}\overline{\delta}} z^{*\alpha} z^{*\beta} a^{*\gamma} a^{*\overline{\delta}} + O(7). \end{aligned}$$

Its inverse mapping is given as follows:

$$\begin{aligned} z^* &= z - \frac{1}{2}p''(0)w^2 + O(6) \\ (2.5) \quad w^* &= w - i\langle z, p''(0) \rangle w^2 - 4iw^2 A_{\alpha\beta\overline{\gamma}\overline{\delta}} z^\alpha z^\beta a^\gamma a^{\overline{\delta}} + O(7). \end{aligned}$$

Then M is transformed by the mapping (2.5) to a real hypersurface up to weight 6 as follows:

$$\begin{aligned} v &= \langle z, z \rangle - 2iu\langle z, p''(0) \rangle \langle z, z \rangle + 2iu\langle p''(0), z \rangle \langle z, z \rangle \\ &\quad - 4iu\langle z, z \rangle A_{\alpha\beta\overline{\gamma}\overline{\delta}} z^\alpha z^\beta a^\gamma a^{\overline{\delta}} + 4iu\langle z, z \rangle A_{\alpha\beta\overline{\gamma}\overline{\delta}} a^\alpha a^\beta z^\gamma z^{\overline{\delta}} \\ &\quad + F_{12}(z, \overline{z}, u) + F_{21}(z, \overline{z}, u) + F_{13}(z, \overline{z}, u) + F_{31}(z, \overline{z}, u) \\ &\quad + F_{22}(z, \overline{z}, u) + F_{23}(z, \overline{z}, u) + F_{32}(z, \overline{z}, u) \\ &\quad + F_{33}(z, \overline{z}, u) + O(z^2 \overline{z}^4) + O(z^4 \overline{z}^2) + O(7) \\ &= \langle z, z \rangle + F_{12}^*(z, \overline{z}, u) + F_{21}^*(z, \overline{z}, u) + F_{13}(z, \overline{z}, u) + F_{31}(z, \overline{z}, u) \\ &\quad + F_{22}(z, \overline{z}, u) + F_{23}^*(z, \overline{z}, u) + F_{32}^*(z, \overline{z}, u) \\ &\quad + F_{33}(z, \overline{z}, u) + O(z^2 \overline{z}^4) + O(z^4 \overline{z}^2) + O(7), \end{aligned}$$

where

$$\begin{aligned} F_{12}^*(z, \overline{z}, u) &= F_{12}(z, \overline{z}, u) + 2iu\langle p''(0), z \rangle \langle z, z \rangle \\ F_{13}^*(z, \overline{z}, u) &= F_{13}(z, \overline{z}, u) + 4i\langle z, z \rangle A_{\alpha\beta\overline{\gamma}\overline{\delta}} a^\alpha a^\beta z^\gamma z^{\overline{\delta}}. \end{aligned}$$

We carry out the following mapping:

$$\begin{aligned}
 z^{*\alpha} &= z^\alpha - 2iz^\alpha \langle z, p''(0) \rangle w + 2g^{\alpha\bar{\delta}} A_{\beta\gamma\bar{\delta}\bar{\eta}} a^{\bar{\eta}} z^\beta z^\gamma w \\
 &\quad + 4ig^{\alpha\bar{\zeta}} A_{\beta\gamma\bar{\eta}\bar{\zeta}} a^{\bar{\eta}} z^\beta z^\gamma \langle z, a \rangle w + 2g^{\alpha\bar{\eta}} C_{\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\beta z^\gamma z^\delta a^{\bar{\xi}} w \\
 &\quad - 8iz^\alpha A_{\beta\gamma\bar{\eta}\bar{\xi}} z^\beta z^\gamma a^{\bar{\eta}} a^{\bar{\xi}} w + O(6) \\
 (2.6) \quad w^* &= w.
 \end{aligned}$$

Then the real hypersurface is transformed to

$$\begin{aligned}
 v &= \langle z, z \rangle + 4u^2 A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^\gamma a^{\bar{\delta}} \\
 &\quad + F_{22}(z, \bar{z}, u) + F_{23}^*(z, \bar{z}, u) + F_{32}^*(z, \bar{z}, u) + F_{33}(z, \bar{z}, u) \\
 &\quad + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + O(7),
 \end{aligned}$$

where

$$F_{23}^*(z, \bar{z}, u) = F_{23}(z, \bar{z}, u) - 2\langle z, z \rangle^2 \langle p''(0), z \rangle + 2i\langle z, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^\gamma a^{\bar{\delta}}.$$

The condition $\Delta^2 F_{23}^* = 0$ determines the coefficient $p''(0)$ so that

$$p''(0) = 0.$$

Then we take a matrix L defined by

$$\langle Lz, z \rangle = 2A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^\gamma a^{\bar{\delta}}.$$

Since $v = \langle z, z \rangle + O(4)$ on the real hypersurface, we have

$$\begin{aligned}
 \langle z, z \rangle + 4u^2 A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^\gamma a^{\bar{\delta}} &= \langle z + u^2 Lz, z + u^2 Lz \rangle + O(7) \\
 &= \langle z + w^2 Lz, z + w^2 Lz \rangle \\
 &\quad + 4\langle z, z \rangle^2 A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^\gamma a^{\bar{\delta}} + O(7).
 \end{aligned}$$

We consider matrices A, B such that

$$\langle z, z \rangle = \langle z + uAz + u^2 Bz, z + uAz + u^2 Bz \rangle + O(7).$$

Since $E = id + O_\times(5)$, we have $A = 0$ so that

$$\begin{aligned}
 \langle z, z \rangle &= \langle z + u^2 Bz, z + u^2 Bz \rangle + O(7) \\
 &= \langle z + w^2 Bz, z + w^2 Bz \rangle - 4iu\langle z, z \rangle \langle Bz, z \rangle + O(7),
 \end{aligned}$$

on the real hypersurface, where

$$\langle Bz, z \rangle + \langle z, Bz \rangle = 0.$$

We carry out the mapping:

$$\begin{aligned}
 z^* &= z + Lzw^2 + Bzw^2 \\
 (2.7) \quad w^* &= w,
 \end{aligned}$$

Then the real hypersurface M is transformed up to weight 6 to

$$\begin{aligned}
 v &= \langle z, z \rangle + F_{22}^*(z, \bar{z}, u) + F_{23}^*(z, \bar{z}, u) + F_{32}^*(z, \bar{z}, u) + F_{33}^*(z, \bar{z}, u) \\
 &\quad + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + O(7),
 \end{aligned}$$

where

$$\begin{aligned}
 F_{22}^*(z, \bar{z}, u) &= F_{22}(z, \bar{z}, u) - 4iu\langle z, z \rangle \langle Bz, z \rangle \\
 F_{33}^*(z, \bar{z}, u) &= F_{33}(z, \bar{z}, u) + 4\langle z, z \rangle^2 A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha a^\beta z^\gamma a^{\bar{\delta}}.
 \end{aligned}$$

The condition $\Delta^2 F_{22} = 0$ determines the matrix B so that

$$\langle Bz, z \rangle = -\frac{3i}{n+2} \{C_{\delta\alpha\beta.\bar{\gamma}}^\delta z^\alpha a^\beta z^\gamma \bar{\gamma} + C_{\delta\alpha.\bar{\beta}\bar{\gamma}}^\delta z^\alpha z^\beta a^\gamma \bar{\gamma}\}.$$

We easily verify $\Delta^3 F_{33}^* = 0$ by using $A_{\alpha\beta.\bar{\delta}}^\alpha = 0$.

Thus we have showed by composing the mappings (2.5), (2.6), (2.7) that the normalization E is given up to weight 6 by the following mapping:

$$\begin{aligned} z^{*\alpha} &= z^\alpha + 2g^{\alpha\bar{\delta}} A_{\beta\gamma\bar{\delta}\bar{\eta}} a^{\bar{\eta}} z^\beta z^\gamma w + 4ig^{\alpha\bar{\zeta}} A_{\beta\gamma\bar{\eta}\bar{\zeta}} a^{\bar{\eta}} z^\beta z^\gamma \langle z, a \rangle w \\ &\quad + 2g^{\alpha\bar{\eta}} C_{\beta\gamma\bar{\delta}\bar{\eta}\bar{\zeta}} z^\beta z^\gamma z^\delta a^\zeta w - 8iz^\alpha A_{\beta\gamma\bar{\eta}\bar{\zeta}} z^\beta z^\gamma a^{\bar{\eta}} a^\zeta w \\ &\quad + 2g^{\alpha\bar{\delta}} A_{\beta\gamma\bar{\delta}\bar{\eta}} z^\beta a^\gamma a^{\bar{\eta}} w^2 \\ &\quad - \frac{3i}{n+2} \cdot g^{\alpha\bar{\delta}} \{C_{\eta\beta\gamma.\bar{\delta}}^\eta z^\beta a^\gamma + C_{\eta\beta.\bar{\delta}\bar{\gamma}}^\eta z^\beta a^\gamma \bar{\gamma}\} w^2 + O(6) \\ w^* &= w - 4iA_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta a^\gamma a^{\bar{\delta}} w^2 + O(7). \end{aligned}$$

This completes the proof. \square

Theorem 2.4. *Let M be a real hypersurface of dimension ≥ 5 in normal form defined by*

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u).$$

Suppose that the origin $0 \in M$ is nonumbilic. Let N_σ be a normalization such that the parameter a in $\sigma = (C, a, \rho, r)$ satisfies the condition:

$$(2.8) \quad \sum_{\alpha} a^\alpha \left(\frac{\partial F_{22}}{\partial z^\alpha} \right) (z, \bar{z}, 0) \neq 0.$$

Then the normalization N_σ satisfies the following condition:

$$(2.9) \quad N_\sigma = \phi_\sigma + O_\times(5) \text{ and } N_\sigma \neq \phi_\sigma + O_\times(6).$$

Conversely, if there is a normalization N_σ satisfying the condition (2.9), then the origin is nonumbilic.

Proof. It may suffice to show that the normalization E in $N_\sigma = \varphi \circ E \circ \psi$ satisfies

$$E = \phi_\sigma + O_\times(5) \text{ and } E \neq \phi_\sigma + O_\times(6)$$

whenever the condition (2.8) is satisfied.

From Lemma 2.3, the normalization E is given up to weight 5 as follows:

$$\begin{aligned} z^{*\alpha} &= z^\alpha + 2g^{\alpha\bar{\delta}} A_{\beta\gamma\bar{\delta}\bar{\eta}} a^{\bar{\eta}} z^\beta z^\gamma w + O(5) \\ w^* &= w + O(6). \end{aligned}$$

Therefore the condition (2.8) implies

$$E = \phi_\sigma + O_\times(5) \text{ and } E \neq \phi_\sigma + O_\times(6)$$

which is equivalent to

$$N_\sigma = \phi_\sigma + O_\times(5) \text{ and } N_\sigma \neq \phi_\sigma + O_\times(6).$$

Clearly, the origin is nonumbilic, i.e.,

$$A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^\gamma \bar{z}^\delta \neq 0$$

whenever there is a normalization N_σ such that

$$N_\sigma = \phi_\sigma + O_\times(5) \text{ and } N_\sigma \neq \phi_\sigma + O_\times(6).$$

This completes the proof. \square

Corollary 2.5 (Webster). *Let M be an analytic real hypersurface of dimension ≥ 5 in a normal coordinate with center at a point $p \in M$ as follows:*

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where the function $F_{22}(z, \bar{z}, u)$ satisfy the following conditions:

$$\Delta^4(F_{22})^2|_0 \neq 0.$$

Then there is a normal coordinate which satisfy the following condition:

$$\Delta^4(F_{22})^2|_0 = \pm 1, \quad \frac{d}{du} \Delta^4(F_{22})^2 \Big|_0 = 0, \quad \Delta^4 \left(F_{22} \frac{\partial F_{23}}{\partial \bar{z}} \right) \Big|_0 = 0.$$

Further, if two real hypersurfaces are of the reduced normal form and they are biholomorphic near the origin, then they are related by a mapping as follows:

$$z^* = Cz, \quad w^* = \pm w,$$

where

$$\langle Cz, Cz \rangle = \pm \langle z, z \rangle.$$

Proof. Let M be defined in a normal coordinate up to weight 5 as follows:

$$\begin{aligned} v &= \langle z, z \rangle + A_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^\gamma z^\delta + C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^\gamma z^\delta z^\eta \\ &\quad + C_{\bar{\alpha}\bar{\beta}\bar{\gamma}\delta\eta} z^{\bar{\alpha}} z^{\bar{\beta}} z^{\bar{\gamma}} z^\delta z^\eta + O(6). \end{aligned}$$

By the mapping $E \circ \psi$, M is mapped up to weight 5 in Theorem 2.4 to

$$\begin{aligned} v &= \langle z, z \rangle + A_{\alpha\beta\bar{\gamma}\bar{\delta}}^* z^\alpha z^\beta z^\gamma z^\delta + C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}}^* z^\alpha z^\beta z^\gamma z^\delta z^\eta \\ &\quad + C_{\bar{\alpha}\bar{\beta}\bar{\gamma}\delta\eta}^* z^{\bar{\alpha}} z^{\bar{\beta}} z^{\bar{\gamma}} z^\delta z^\eta + O(6), \end{aligned}$$

where

$$\begin{aligned} A_{\alpha\beta\bar{\gamma}\bar{\delta}}^* &= A_{\alpha\beta\bar{\gamma}\bar{\delta}}, \\ C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}}^* &= C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} - \frac{i}{3} \left\{ g_{\alpha\bar{\delta}} A_{\beta\gamma\bar{\eta}\bar{\xi}} a^{\bar{\xi}} + g_{\beta\bar{\delta}} A_{\gamma\alpha\bar{\eta}\bar{\xi}} a^{\bar{\xi}} + g_{\gamma\bar{\delta}} A_{\alpha\beta\bar{\eta}\bar{\xi}} a^{\bar{\xi}} \right. \\ &\quad \left. + g_{\alpha\bar{\eta}} A_{\beta\gamma\bar{\delta}\bar{\xi}} a^{\bar{\xi}} + g_{\beta\bar{\eta}} A_{\gamma\alpha\bar{\delta}\bar{\xi}} a^{\bar{\xi}} + g_{\gamma\bar{\eta}} A_{\alpha\beta\bar{\delta}\bar{\xi}} a^{\bar{\xi}} \right\}. \end{aligned}$$

Then we obtain

$$(2.10) \quad A_{\alpha\beta\bar{\gamma}\bar{\delta}}^* C_{\gamma\delta\bar{\zeta}\bar{\eta}}^* = A_{\alpha\beta\bar{\gamma}\bar{\delta}}^* C_{\gamma\delta\bar{\zeta}\bar{\eta}}^* - \frac{2i}{3} (A_{\alpha\beta\bar{\gamma}\bar{\delta}}^* A_{\gamma\delta\bar{\zeta}\bar{\eta}}^*) g_{\zeta\bar{\eta}} a^{\bar{\eta}},$$

where we raise indices by using the following matrix:

$$(g_{\beta\bar{\delta}}) = \text{diag} \{ \underbrace{1, \dots, 1}_e, \underbrace{-1, \dots, -1}_{n-e} \}.$$

Suppose that

$$\begin{aligned} F_{22}(z, \bar{z}, u) &= N_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta z^\gamma z^\delta \\ F_{23}(z, \bar{z}, u) &= N_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^\gamma z^\delta z^\eta. \end{aligned}$$

Since $\Delta F_{22} = 0$, we obtain

$$\begin{aligned}\Delta^4(F_{22})^2 &= 3 \cdot 2^5 N_{\alpha\beta..}^{\gamma\delta} N_{\gamma\delta..}^{\alpha\beta} \\ \Delta^4\left(F_{22} \frac{\partial F_{23}}{\partial \bar{z}}\right) &= 9 \cdot 2^5 N_{\alpha\beta..}^{\gamma\delta} N_{\gamma\delta..\bar{\zeta}}^{\alpha\beta}.\end{aligned}$$

Thus the condition equality (2.10) reads

$$\Delta^4\left(F_{22}^* \frac{\partial F_{23}^*}{\partial \bar{z}}\right)\Big|_0 = \Delta^4\left(F_{22} \frac{\partial F_{23}}{\partial \bar{z}}\right)\Big|_0 + 2ia \Delta^4(F_{22})^2\Big|_0.$$

We take a normalization N_σ with

$$\begin{aligned}\sigma &= (id_{n \times n}, a, 1, 0) \\ a &= \frac{i \Delta^4\left(F_{22} \frac{\partial F_{23}}{\partial \bar{z}}\right)\Big|_0}{2 \Delta^4(F_{22})^2\Big|_0}\end{aligned}$$

so that

$$\Delta^4\left(F_{22}^* \frac{\partial F_{23}^*}{\partial \bar{z}}\right)\Big|_0 = 0.$$

Note that $a = 0$ necessarily if

$$\Delta^4\left(F_{22} \frac{\partial F_{23}}{\partial \bar{z}}\right)\Big|_0 = 0.$$

Then we carry out a normalization N_σ with $\sigma = (\sqrt{\rho}, 0, \rho, r)$, $\rho > 0$:

$$z^* = \frac{\sqrt{\rho}z}{1 - rw}, \quad w^* = \frac{\rho w}{1 - rw}$$

so that

$$\begin{aligned}F_{22}^*(z, \bar{z}, 0) &= \rho^{-1} F_{22}(z, \bar{z}, 0) \\ \left(\frac{\partial F_{22}^*}{\partial u}\right)(z, \bar{z}, 0) &= \rho^{-1} \left(\frac{\partial F_{22}}{\partial u}\right)(z, \bar{z}, 0) - 2\rho^{-2} r F_{22}(z, \bar{z}, 0).\end{aligned}$$

Then we obtain

$$\begin{aligned}\Delta^4(F_{22}^*)^2\Big|_0 &= \rho^{-2} \Delta^4(F_{22})^2\Big|_0 \\ \frac{d}{du} \Delta^4(F_{22}^*)^2\Big|_0 &= \rho^{-2} \frac{d}{du} \Delta^4(F_{22})^2\Big|_0 - 4\rho^{-3} r \Delta^4(F_{22})^2\Big|_0.\end{aligned}$$

We take

$$\begin{aligned}\rho &= \sqrt{|\Delta^4(F_{22})^2|}\Big|_0 \\ r &= \text{sign}\{\Delta^4(F_{22})^2\Big|_0\} \frac{\frac{d}{du} \Delta^4(F_{22})^2\Big|_0}{4 \left(\sqrt{|\Delta^4(F_{22})^2|}\Big|_0\right)^3}\end{aligned}$$

so that

$$\begin{aligned}\Delta^4(F_{22}^*)^2\Big|_0 &= \pm 1 \\ \frac{d}{du} \Delta^4(F_{22}^*)^2\Big|_0 &= 0.\end{aligned}$$

Suppose that M, M' are in reduced normal form by the conditions

$$\Delta^4 \left(F_{22} \frac{\partial F_{23}}{\partial \bar{z}} \right) \Big|_0 = 0, \quad \Delta^4 (F_{22})^2 \Big|_0 = \pm 1, \quad \frac{d}{du} \Delta^4 (F_{22})^2 \Big|_0 = 0$$

and there is a normalization of M, N_σ , with $\sigma = (C, a, \rho, r)$ satisfying $M' = N_\sigma(M)$. Then necessarily we have

$$a = 0, \quad \rho = \pm 1, \quad r = 0$$

so that

$$N_\sigma : \begin{cases} z^* = Cz, \\ w^* = \pm w, \end{cases}$$

where

$$\langle Cz, Cz \rangle = \pm \langle z, z \rangle.$$

This completes the proof. \square

3. SPHERICAL ANALYTIC REAL HYPERSURFACES

I. We shall study a nondegenerate analytic real hypersurface M with an open subset of umbilic points.

Lemma 3.1. *Let k be a positive integer ≥ 7 . Suppose that M is a real hypersurface in normal form such that*

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, u) + \sum_{\min(s,t) \geq 2, s+t \geq k+1} F_{st}(z, \bar{z}, u),$$

where

$$\sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, 0) \neq 0.$$

Then there is a vector $a \in \mathbb{C}^n$ and a normalization of M, N_σ , $\sigma = (id_{n \times n}, a, 1, 0)$, such that

$$F^*(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2}^{s+t=k-1} F_{st}^*(z, \bar{z}, u) + \sum_{\min(s,t) \geq 2, s+t \geq k} F_{st}^*(z, \bar{z}, u),$$

where the real hypersurface $N_\sigma(M)$ is defined by

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u)$$

and

$$\sum_{\min(s,t) \geq 2, s+t=k-1} F_{st}^*(z, \bar{z}, u) \neq 0.$$

Proof. The real hypersurface M is defined up to weight k as follows:

$$\begin{aligned} v &= \langle z, z \rangle + \sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, 0) \\ &+ \sum_{\min(s,t) \geq 2, s+t=k+1} O(z^s \bar{z}^t) + O(k+2). \end{aligned}$$

By the mapping ψ in the decomposition of $N_\sigma = \varphi \circ E \circ \psi$, we obtain

$$\begin{aligned}
v &= \langle z, z \rangle \\
&+ \sum_{\min(s,t) \geq 2, s+t=k} \sum_{\alpha} u \left\{ a^\alpha \left(\frac{\partial F_{st}}{\partial z^\alpha} \right) (z, \bar{z}, 0) + \bar{a}^\alpha \left(\frac{\partial F_{st}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, 0) \right\} \\
&+ \sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, 0) + \sum_{\min(s,t) \geq 2, s+t=k+1} O(z^s \bar{z}^t) + O(k+2) \\
&= \langle z, z \rangle \\
&+ \sum_{\alpha} u \left\{ a^\alpha \left(\frac{\partial F_{2,k-2}}{\partial z^\alpha} \right) (z, \bar{z}, 0) + \bar{a}^\alpha \left(\frac{\partial F_{k-2,2}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, 0) \right\} \\
&+ \sum_{\min(s,t) \geq 2, s+t=k-1} \sum_{\alpha} u \left\{ a^\alpha \left(\frac{\partial F_{s+1,t}}{\partial z^\alpha} \right) (z, \bar{z}, 0) \right. \\
&\quad \left. + \bar{a}^\alpha \left(\frac{\partial F_{s,t+1}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, 0) \right\} \\
&+ \sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, 0) + \sum_{\min(s,t) \geq 2, s+t=k+1} O(z^s \bar{z}^t) + O(k+2).
\end{aligned}$$

Then by the normalization E , we obtain for $k \geq 8$

$$\begin{aligned}
v &= \langle z, z \rangle \\
&+ \sum_{\min(s,t) \geq 2, s+t=k-1} \sum_{\alpha} u \left\{ a^\alpha \left(\frac{\partial F_{s+1,t}}{\partial z^\alpha} \right) (z, \bar{z}, 0) \right. \\
&\quad \left. + \bar{a}^\alpha \left(\frac{\partial F_{s,t+1}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, 0) \right\} \\
&+ \sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, 0) + \sum_{\min(s,t) \geq 2, s+t=k+1} O(z^s \bar{z}^t) + O(k+2).
\end{aligned}$$

For $k = 7$, we take a vector $a = (a^\alpha) \in \mathbb{C}^n$ so that

$$\begin{aligned}
\left(\frac{\partial F_{42}^*}{\partial u} \right) (z, \bar{z}, 0) &= \sum_{\alpha} \left\{ a^\alpha \left(\frac{\partial F_{52}}{\partial z^\alpha} \right) (z, \bar{z}, 0) + \bar{a}^\alpha \left(\frac{\partial F_{43}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, 0) \right\} \\
(3.1) \quad &\neq 0.
\end{aligned}$$

By the normalization E , we obtain for $k = 7$

$$\begin{aligned}
v &= \langle z, z \rangle \\
&+ \sum_{\min(s,t) \geq 2, s+t=6} \sum_{\alpha} u \left\{ a^\alpha \left(\frac{\partial F_{s+1,t}}{\partial z^\alpha} \right) (z, \bar{z}, 0) \right. \\
&\quad \left. + \bar{a}^\alpha \left(\frac{\partial F_{s,t+1}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, 0) \right\} \\
&+ \kappa \langle z, z \rangle^3 + \sum_{\min(s,t) \geq 2, s+t=7} F_{st}(z, \bar{z}, 0) \\
&+ \sum_{\min(s,t) \geq 2, s+t=8} O(z^s \bar{z}^t) + O(9)
\end{aligned}$$

where the constant κ is determined so that $\Delta^3 F_{33}^* = 0$.

We easily see that the condition (3.1) remains valid for $k = 7$. Thus, for $k \geq 7$, we have showed that there is a vector $a \in \mathbb{C}^n$ such that

$$\sum_{\min(s,t) \geq 2, s+t=k-1} \left(\frac{\partial F_{st}^*}{\partial u} \right) (z, \bar{z}, 0) u \neq 0$$

This completes the proof. \square

Lemma 3.2. *Let M be a real hypersurface in normal form, which is defined up to weight 6 by the following equation:*

$$\begin{aligned} v = \langle z, z \rangle + A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi + A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi \\ + A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi + O(7) \end{aligned}$$

where

$$\overline{A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}}} = A_{\eta\xi\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}, \quad \overline{A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}}} = A_{\delta\eta\xi\bar{\alpha}\bar{\beta}\bar{\gamma}}, \quad A_{\alpha\beta\gamma\dots}^{\alpha\beta\gamma} = 0.$$

Let N_σ be a normalization of M with initial value $\sigma = (id_{n \times n}, a, 1, 0)$. Suppose that $N_\sigma(M)$ is defined by the equation

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}^*(z, \bar{z}, u).$$

Then the function $F_{32}^*(z, \bar{z}, u)$ is given by

$$\begin{aligned} & F_{32}^*(z, \bar{z}, u) \\ &= 4u A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma a^\delta z^\eta z^\xi + 3u A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta a^\xi \\ & \quad - \frac{6u \langle z, z \rangle^2}{(n+1)(n+2)} \{ 4A_{\alpha\beta\gamma\delta\dots}^{\alpha\beta} z^\gamma a^\delta + 3A_{\alpha\beta\gamma\delta\dots}^{\alpha\beta} z^\gamma a^\delta \}. \end{aligned}$$

Proof. Suppose that M is given up to weight 7 by the following equation:

$$\begin{aligned} v = & \langle z, z \rangle + A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi + A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi \\ & + A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi \\ & + C_{\alpha\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi z^\rho + C_{\alpha\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi z^\rho \\ & + C_{\alpha\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi z^\rho + C_{\alpha\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\xi z^\rho \\ & + \sum_{\min(s,t) \geq 2, s+t=8} O(z^s \bar{z}^t) + \sum_{\min(s,t) \geq 2, s+t=6} O(z^s \bar{z}^t u) + O(9), \end{aligned}$$

where

$$\begin{aligned} \overline{A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}}} &= A_{\eta\xi\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}, \quad \overline{A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}}} = A_{\delta\eta\xi\bar{\alpha}\bar{\beta}\bar{\gamma}}, \\ \overline{C_{\alpha\beta\gamma\delta\eta\bar{\xi}\bar{\rho}}} &= C_{\xi\rho\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}\bar{\eta}}, \quad \overline{C_{\alpha\beta\gamma\delta\eta\bar{\xi}\bar{\rho}}} = C_{\eta\xi\rho\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}, \\ A_{\alpha\beta\gamma\dots}^{\alpha\beta\gamma} &= 0, \end{aligned}$$

and all barred and unbarred indices are respectively symmetric.

By the mapping ψ in the decomposition of $N_\sigma = \varphi \circ E \circ \psi$, M is transformed up to weight 7 to a real hypersurface as follows:

$$\begin{aligned} v &= \langle z, z \rangle + F_{04}(z, \bar{z}, u) + F_{40}(z, \bar{z}, u) \\ &\quad + F_{13}(z, \bar{z}, u) + F_{31}(z, \bar{z}, u) + F_{22}(z, \bar{z}, u) \\ &\quad + F_{14}(z, \bar{z}, u) + F_{41}(z, \bar{z}, u) + F_{23}(z, \bar{z}, u) + F_{32}(z, \bar{z}, u) \\ &\quad + F_{15}(z, \bar{z}, u) + F_{51}(z, \bar{z}, u) + F_{33}(z, \bar{z}, u) \\ &\quad + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + \sum_{\min(s,t) \geq 2, s+t=7} O(z^s \bar{z}^t) + O(9), \end{aligned}$$

where

$$\begin{aligned} F_{40}(z, \bar{z}, u) &= 2u^2 A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{a}^\eta \bar{a}^\xi \\ F_{31}(z, \bar{z}, u) &= 8u^2 A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma a^\delta \bar{z}^\eta \bar{a}^\xi + 6u^2 A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{a}^\eta \bar{a}^\xi \\ F_{41}(z, \bar{z}, u) &= 2u A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{z}^\eta \bar{a}^\xi \\ F_{51}(z, \bar{z}, u) &= -4iu \langle z, z \rangle A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{a}^\eta \bar{a}^\xi + 2u C_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}\bar{\rho}} z^\alpha z^\beta z^\gamma z^\delta z^\eta \bar{z}^\xi \bar{a}^\rho \\ &\quad + 12iu \langle z, a \rangle A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{z}^\eta \bar{a}^\xi \\ F_{22}(z, \bar{z}, u) &= 12u^2 \{ A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta a^\gamma a^\delta \bar{z}^\eta \bar{z}^\xi + A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{a}^\eta \bar{a}^\xi \} \\ &\quad + 9u^2 A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta a^\gamma z^\delta \bar{z}^\eta \bar{a}^\xi \\ F_{32}(z, \bar{z}, u) &= 4u A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma a^\delta \bar{z}^\eta \bar{z}^\xi + 3u A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{z}^\eta \bar{a}^\xi \\ F_{33}(z, \bar{z}, u) &= 24iu \langle z, z \rangle \{ A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta a^\gamma a^\delta \bar{z}^\eta \bar{z}^\xi - A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{a}^\eta \bar{a}^\xi \} \\ &\quad + 8iu \left\{ \langle z, a \rangle A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{z}^\eta \bar{a}^\xi - \right. \\ &\quad \left. \langle a, z \rangle A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma a^\delta \bar{z}^\eta \bar{z}^\xi \right\} \\ &\quad + A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{z}^\eta \bar{z}^\xi \\ &\quad + 12iu \left\{ \langle z, a \rangle A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta a^\gamma z^\delta \bar{z}^\eta \bar{z}^\xi - \right. \\ &\quad \left. \langle a, z \rangle A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta \bar{z}^\eta \bar{a}^\xi \right\} \\ &\quad + 4u \{ C_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}\bar{\rho}} z^\alpha z^\beta z^\gamma a^\delta \bar{z}^\eta \bar{z}^\xi \bar{a}^\rho + C_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}\bar{\rho}} z^\alpha z^\beta z^\gamma z^\delta \bar{z}^\eta \bar{z}^\xi \bar{a}^\rho \}. \end{aligned}$$

Since M is in normal form, $E = id + O_\times(7)$ by Theorem 1.5. So the function $p(u)$ satisfies

$$p(u) = \frac{1}{6} p'''(0) u^3 + O(8).$$

Let $g(z, w)$ be a holomorphic function(cf. Lemma 3 in the paper [Pa2]) defined by

$$\begin{aligned} g(z, w) - g(0, w) &= -2iF(p(w), \bar{p}(w), w) \\ &\quad + 2iF\left(z + p(w), \bar{p}(w), w + \frac{1}{2}\{g(z, w) - g(0, w)\}\right) \\ g(0, w) &= iF(p(w), \bar{p}(w), w). \end{aligned}$$

Thus the function $g(0, u)$ satisfies

$$g(0, u) = iF(p(u), \bar{p}(u), u) = O(12).$$

Hence the holomorphic function $g(z, w)$ is implicitly defined up to weight 11 as follows:

$$g = 2iF\left(z + p(w), \bar{p}(w), w + \frac{1}{2}g\right) + O(12).$$

Thus we obtain

$$\begin{aligned} g &= 2iF\left(z + p(w), \bar{p}(w), w + \frac{1}{2}g\right) + O(12) \\ &= \frac{i}{3}\langle z, p'''(0) \rangle w^3 + 2iF\left(z, 0, w + \frac{1}{2}g\right) + O(9) \\ &= \frac{i}{3}\langle z, p'''(0) \rangle w^3 + 4i(w + \frac{1}{2}g)^2 A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta a^\eta \bar{a}^\xi + O(9) \\ &= \frac{i}{3}\langle z, p'''(0) \rangle w^3 + 4iw^2 A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta a^\eta \bar{a}^\xi + O(9). \end{aligned}$$

We carry out the following mapping:

$$\begin{aligned} z &= z^* + \frac{1}{6}p'''(0)w^{*3} + O(8) \\ w &= w^* + \frac{i}{3}\langle z^*, p'''(0) \rangle w^{*3} + 4iw^{*2} A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^{*\alpha} z^{*\beta} z^{*\gamma} z^{*\delta} a^\eta \bar{a}^\xi + O(9). \end{aligned}$$

Its inverse mapping is given as follows:

$$(3.2) \quad \begin{aligned} z^* &= z - \frac{1}{6}p'''(0)w^3 + O(8) \\ w^* &= w - \frac{i}{3}\langle z, p'''(0) \rangle w^3 - 4iw^2 A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta a^\eta \bar{a}^\xi + O(9). \end{aligned}$$

Then M is transformed by the mapping (3.2) to a real hypersurface up to weight 6 as follows:

$$\begin{aligned} v &= \langle z, z \rangle - iu^2 \langle z, z \rangle \langle z, p'''(0) \rangle + iu^2 \langle z, z \rangle \langle p'''(0), z \rangle \\ &\quad - 4iu \langle z, z \rangle A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta a^\eta \bar{a}^\xi + 4iu \langle z, z \rangle A_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}\bar{\xi}} a^\alpha a^\beta z^\gamma z^\delta \bar{a}^\eta \bar{a}^\xi \\ &\quad + F_{13}(z, \bar{z}, u) + F_{31}(z, \bar{z}, u) + F_{22}(z, \bar{z}, u) \\ &\quad + F_{14}(z, \bar{z}, u) + F_{41}(z, \bar{z}, u) + F_{23}(z, \bar{z}, u) + F_{32}(z, \bar{z}, u) \\ &\quad + F_{15}(z, \bar{z}, u) + F_{51}(z, \bar{z}, u) + F_{33}(z, \bar{z}, u) \\ &\quad + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + \sum_{\min(s,t) \geq 2, s+t=7} O(z^s \bar{z}^t) + O(9) \\ &= \langle z, z \rangle + F_{12}(z, \bar{z}, u) + F_{21}(z, \bar{z}, u) + F_{13}(z, \bar{z}, u) + F_{31}(z, \bar{z}, u) \\ &\quad + F_{22}(z, \bar{z}, u) + F_{14}(z, \bar{z}, u) + F_{41}(z, \bar{z}, u) + F_{23}(z, \bar{z}, u) \\ &\quad + F_{32}(z, \bar{z}, u) + F_{15}^*(z, \bar{z}, u) + F_{51}^*(z, \bar{z}, u) + F_{33}(z, \bar{z}, u) \\ &\quad + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + \sum_{\min(s,t) \geq 2, s+t=7} O(z^s \bar{z}^t) + O(9), \end{aligned}$$

where

$$\begin{aligned} F_{21}^*(z, \bar{z}, u) &= -iu^2 \langle z, z \rangle \langle z, p'''(0) \rangle \\ F_{51}^*(z, \bar{z}, u) &= F_{51}(z, \bar{z}, u) - 4iu \langle z, z \rangle A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta a^\eta \bar{a}^\xi. \end{aligned}$$

We carry out the following mapping:

$$\begin{aligned}
z^{*\alpha} &= z^\alpha - iz^\alpha \langle z, p'''(0) \rangle w^2 + 8g^{\alpha\bar{\xi}} A_{\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\beta z^\gamma z^\delta a^\eta \bar{a}^\rho w^2 \\
&\quad + 6g^{\alpha\bar{\xi}} A_{\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\beta z^\gamma z^\delta \bar{a}^\eta \bar{a}^\rho w^2 \\
&\quad + 2g^{\alpha\bar{\xi}} A_{\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\beta z^\gamma z^\delta z^\eta \bar{a}^\rho w - 8iz^\alpha A_{\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\beta z^\gamma z^\delta z^\eta \bar{a}^\xi \bar{a}^\rho w \\
&\quad + 12i \langle z, a \rangle g^{\alpha\bar{\xi}} A_{\beta\gamma\delta\eta\bar{\xi}\bar{\rho}} z^\beta z^\gamma z^\delta z^\eta \bar{a}^\rho w + 2g^{\alpha\bar{\rho}} C_{\beta\gamma\delta\eta\bar{\xi}\bar{\rho}\bar{\sigma}} z^\beta z^\gamma z^\delta z^\eta z^\xi \bar{a}^\sigma w \\
&\quad + O(8) \\
w^* &= w.
\end{aligned}$$

Then the real hypersurface is transformed to

$$\begin{aligned}
v &= \langle z, z \rangle - 2u \langle z, z \rangle^2 \langle z, p'''(0) \rangle - 2u \langle z, z \rangle^2 \langle p'''(0), z \rangle \\
&\quad + F_{22}(z, \bar{z}, u) + F_{23}(z, \bar{z}, u) + F_{32}(z, \bar{z}, u) + F_{33}(z, \bar{z}, u) \\
&\quad + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + \sum_{\min(s,t) \geq 2, s+t=7} O(z^s \bar{z}^t) + O(9) \\
&= \langle z, z \rangle + F_{22}(z, \bar{z}, u) + F_{23}^*(z, \bar{z}, u) + F_{32}^*(z, \bar{z}, u) + F_{33}(z, \bar{z}, u) \\
&\quad + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + \sum_{\min(s,t) \geq 2, s+t=7} O(z^s \bar{z}^t) + O(9),
\end{aligned}$$

where

$$\begin{aligned}
F_{32}^*(z, \bar{z}, u) &= F_{32}(z, \bar{z}, u) - 2u \langle z, z \rangle^2 \langle z, p'''(0) \rangle \\
&\quad = 4u A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma a^\delta \bar{z}^\eta \bar{z}^\xi + 3u A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma \bar{z}^\delta \bar{z}^\eta \bar{a}^\xi \\
(3.3) \quad &\quad - 2u \langle z, z \rangle^2 \langle z, p'''(0) \rangle.
\end{aligned}$$

The condition $\Delta^2 F_{23}^* = 0$ determines the coefficient $p'''(0)$ so that

$$(3.4) \quad \langle z, p'''(0) \rangle = \frac{3}{(n+1)(n+2)} \{ 4A_{\alpha\beta\gamma\delta..}^{\alpha\beta} z^\gamma a^\delta + 3A_{\alpha\beta\gamma\delta..}^{\alpha\beta} z^\gamma \bar{a}^\delta \}.$$

Since $E = id + O_\times(7)$, we consider a matrix B such that

$$\langle z, z \rangle = \langle z + u^3 Bz, z + u^3 Bz \rangle + O(9),$$

which is equivalent to the following condition:

$$\langle Bz, z \rangle + \langle z, Bz \rangle = 0.$$

Hence, on the real hypersurface, we have

$$\begin{aligned}
\langle z, z \rangle &= \langle z + u^3 Bz, z + u^3 Bz \rangle + O(9) \\
&= \langle z + w^3 Bz, z + w^3 Bz \rangle - 6iu^2 \langle z, z \rangle \langle Bz, z \rangle + 2i \langle z, z \rangle^3 \langle Bz, z \rangle \\
&\quad + O(9).
\end{aligned}$$

We carry out the mapping:

$$\begin{aligned}
z^* &= z + Bzw^3 \\
w^* &= w,
\end{aligned}$$

Then the real hypersurface M is transformed up to weight 8 to

$$\begin{aligned}
v &= \langle z, z \rangle + F_{22}^*(z, \bar{z}, u) + F_{23}^*(z, \bar{z}, u) + F_{32}^*(z, \bar{z}, u) + F_{33}(z, \bar{z}, u) \\
&\quad + O(z^2 \bar{z}^4) + O(z^4 \bar{z}^2) + \sum_{\min(s,t) \geq 2, s+t=7} O(z^s \bar{z}^t) + O(9),
\end{aligned}$$

where

$$F_{22}^*(z, \bar{z}, u) = F_{22}(z, \bar{z}, u) - 6iu^2 \langle z, z \rangle \langle Bz, z \rangle.$$

The condition $\Delta^2 F_{22}^* = 0$ determines the matrix B so that

$$\begin{aligned} \langle Bz, z \rangle &= -\frac{2i}{n+2} \left\{ 4A_{\rho\beta\gamma\delta.\bar{\xi}}^{\rho} z^{\beta} a^{\gamma} a^{\delta} z^{\bar{\xi}} + 4A_{\rho\beta.\bar{\delta}\bar{\eta}\bar{\xi}}^{\rho} z^{\beta} z^{\bar{\delta}} a^{\bar{\eta}} a^{\bar{\xi}} + \right. \\ &\quad \left. 3A_{\rho\beta\gamma.\bar{\eta}\bar{\xi}}^{\rho} z^{\beta} a^{\gamma} z^{\bar{\eta}} a^{\bar{\xi}} \right\} \\ &\quad + \frac{i\langle z, z \rangle}{(n+1)(n+2)} \left\{ 4A_{\rho\sigma\gamma\delta..}^{\rho\sigma} a^{\gamma} a^{\delta} + 4A_{\rho\sigma\beta..\bar{\eta}\bar{\xi}}^{\rho\sigma} a^{\bar{\eta}} a^{\bar{\xi}} + \right. \\ &\quad \left. 3A_{\rho\sigma\gamma..\bar{\xi}}^{\rho\sigma} a^{\gamma} a^{\bar{\xi}} \right\}. \end{aligned}$$

Then we obtain $\Delta^3 F_{33}^* = 0$ by putting

$$F_{33}^*(z, \bar{z}, u) = F_{33}(z, \bar{z}, u) - \kappa u \langle z, z \rangle^3$$

where

$$\begin{aligned} \kappa &= \frac{48i}{n(n+2)} \left\{ A_{\rho\sigma\gamma\delta..}^{\rho\sigma} a^{\gamma} a^{\delta} - A_{\rho\sigma..\bar{\eta}\bar{\xi}}^{\rho\sigma} a^{\bar{\eta}} a^{\bar{\xi}} \right\} \\ &\quad + \frac{24}{n(n+1)(n+2)} \left\{ C_{\alpha\beta\gamma\delta...}^{\alpha\beta\gamma} a^{\delta} + C_{\alpha\beta\gamma...\bar{\rho}}^{\alpha\beta\gamma} a^{\bar{\rho}} \right\}. \end{aligned}$$

Thus we have completed the normalizing process up to weight 8. Therefore, the desired result is obtained by the equalities (3.3) and (3.4). This completes the proof. \square

II. Main theorem

Theorem 3.3. *Let M be a nondegenerate analytic real hypersurface in a complex manifold and U be an open subset of M consisting of umbilic points. Then the open subset U is locally biholomorphic to a real hyperquadric.*

Proof. For the case of $n = 1$, we have $F_{22} = F_{23} = F_{33} = 0$. By the definition on umbilic points for $n = 1$ and invariance of normal form under translation along u -curve, in any normal coordinate at any point of U we have

$$F_{24} = F_{42} = 0.$$

Suppose that there is a positive integer $k \geq 7$ and a normal coordinate at a point p in U such that

$$(3.5) \quad v = z\bar{z} + \sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, u) + \sum_{\min(s,t) \geq 2, s+t \geq k+1} F_{st}(z, \bar{z}, u),$$

where

$$\sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, u) \neq 0.$$

Let's take a point $p \in U$ and a normal coordinate so that k is the smallest integer satisfying the condition in (3.5). Since the normal form is invariant under translation along the u -curve, we can assume that

$$\sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, 0) \neq 0.$$

By Lemma 3.1 this is a contradiction to the choice of the integer k . Therefore M is defined by

$$v = z\bar{z}$$

in any normal coordinate at any point in U . This completes the proof for $n = 1$.

For the case of $n \geq 2$, by Lemma 3.1 it suffices to show that

$$F_{22} = F_{23} = F_{24} = F_{33} = 0.$$

Since the normal form is invariant under translation along u -curve, we have

$$F_{22} = 0$$

in any normal coordinate at any point in U .

Suppose that there is a point $p \in U$ and a normal coordinate at p such that $F_{23} \neq 0$. Further, without loss of generality we can assume that $F_{23}(z, \bar{z}, 0) \neq 0$ so that

$$F_{23}(z, \bar{z}, 0) = C_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^\gamma z^\delta z^\eta \bar{z} \neq 0.$$

Then from the proof of Theorem 2.4 we obtain the following identity for all $a \in \mathbb{C}^n$

$$\begin{aligned} F_{22}^*(z, \bar{z}, u) &= 3uC_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} z^\alpha z^\beta a^\gamma z^\delta z^\eta + 3uC_{\alpha\beta\bar{\gamma}\delta\bar{\eta}} z^\alpha z^\beta z^\gamma z^\delta a^\eta \\ &\quad - \frac{12}{n+2} \cdot u\langle z, z \rangle \left\{ C_{\delta\alpha\beta.\bar{\gamma}}^\delta z^\alpha a^\beta z^\gamma + C_{\delta\alpha.\bar{\beta}\bar{\gamma}}^\delta z^\alpha z^\beta a^\gamma \right\} \\ &= 0. \end{aligned}$$

Hence we obtain

$$(n+2)C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} = h_{\alpha\bar{\delta}} C_{\rho\beta\gamma.\bar{\eta}}^\rho + h_{\beta\bar{\delta}} C_{\rho\alpha\gamma.\bar{\eta}}^\rho + h_{\alpha\bar{\eta}} C_{\rho\beta\gamma.\bar{\delta}}^\rho + h_{\beta\bar{\eta}} C_{\rho\alpha\gamma.\bar{\delta}}^\rho.$$

Contracting the pair $(\gamma, \bar{\delta})$ yields

$$C_{\rho\beta\gamma.\bar{\eta}}^\rho = 0,$$

which implies

$$C_{\alpha\beta\gamma\bar{\delta}\bar{\eta}} = 0.$$

This is a contradiction to the hypothesis that $F_{23}(z, \bar{z}, 0) = C_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}} z^\alpha z^\beta z^\gamma z^\delta z^\eta \bar{z} \neq 0$. Thus

$$F_{23} = F_{32} = 0$$

in any normal coordinate at any point in U .

Suppose that there is a point $p \in U$ and a normal coordinate at the point such that

$$F_{24} + F_{33} + F_{42} \neq 0.$$

Without loss of generality, we may assume

$$F_{24}(z, \bar{z}, 0) \neq 0 \quad \text{or} \quad F_{33}(z, \bar{z}, 0) \neq 0.$$

Let's put

$$\begin{aligned} F_{24}(z, \bar{z}, 0) &= A_{\alpha\beta\bar{\gamma}\bar{\delta}\bar{\eta}\bar{\sigma}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\sigma \bar{z} \\ F_{33}(z, \bar{z}, 0) &= A_{\alpha\beta\gamma\bar{\delta}\bar{\eta}\bar{\sigma}} z^\alpha z^\beta z^\gamma z^\delta z^\eta z^\sigma \bar{z}. \end{aligned}$$

Since $F_{32}^*(z, \bar{z}, u) = 0$ identically, Lemma 3.2 yields the following identity for all $a \in \mathbb{C}^n$:

$$\begin{aligned} & 4A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma a^\delta z^\eta z^\xi + 3A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta a^\xi \\ &= \frac{6\langle z, z \rangle^2}{(n+1)(n+2)} \left\{ 4A_{\alpha\beta\gamma\delta..}^{\alpha\beta} z^\gamma a^\delta + 3A_{\alpha\beta\gamma\delta..}^{\alpha\beta} z^\gamma a^\delta \right\}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma a^\delta z^\eta z^\xi &= \frac{6}{(n+1)(n+2)} \langle z, z \rangle^2 A_{\alpha\beta\gamma\delta..}^{\alpha\beta} z^\gamma a^\delta, \\ A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} z^\alpha z^\beta z^\gamma z^\delta z^\eta a^\xi &= \frac{6}{(n+1)(n+2)} \langle z, z \rangle^2 A_{\alpha\beta\gamma\delta..}^{\alpha\beta} z^\gamma a^\delta. \end{aligned}$$

Putting $a^\alpha = z^\alpha$ and computing the derivative Δ^2 yields

$$A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} = A_{\alpha\beta\gamma\delta\bar{\eta}\bar{\xi}} = 0.$$

This is a contradiction to the hypothesis

$$F_{24}(z, \bar{z}, 0) \neq 0 \quad \text{or} \quad F_{33}(z, \bar{z}, 0) \neq 0.$$

So $F_{24} = F_{42} = F_{33} = 0$ in any normal coordinate at any point in U .

Then, by Lemma 3.1, M is defined by

$$v = \langle z, z \rangle$$

in any normal coordinate at any point in U . This completes the proof. \square

There is a global version of Theorem 0.4. We shall give a complete proof of this global version to the paper [Pa3], where we present a new proof of the following theorem:

Lemma 3.4 (Pinchuk). *Let M be a nondegenerate analytic real hypersurface. Suppose that there are two points p, q on M and a curve γ on M connecting the two points p, q . Then p is a spherical point if and only if q is a spherical point.*

From Lemma 3.4, we obtain the following global result:

Lemma 3.5. *Let M be a connected nondegenerate analytic real hypersurface. Suppose that the subset of umbilic points of M has an interior point. Then M is locally biholomorphic to a real hyperquadric at every point of M .*

Then we obtain the following theorem from Theorem 3.3 and Lemma 3.5

Theorem 3.6. *Let M be a connected nondegenerate analytic real hypersurface in a complex manifold. Suppose that there is a point p on M at which M is not locally biholomorphic to a real hyperquadric. Then M is not locally biholomorphic to a real hyperquadric at every point of M .*

III. A family of nondegenerate analytic real hypersurfaces M parametrized by μ . From Theorem 1.4, we obtain

Lemma 3.7. *Let M_μ be a nondegenerate analytic real hypersurface defined by*

$$v = F(z, \bar{z}, u; \mu), \quad F|_0 = dF|_0 = 0,$$

where μ is a real parameter and $F(z, \bar{z}, u; \mu)$ is real analytic of μ near $\mu = 0$. Let $\phi = (f, g)$ be a normalization of M_μ with initial value independent of μ . Suppose that $\phi(M_\mu)$ is defined by $v = \langle z, z \rangle + F^*(z, \bar{z}, u)$. Then the functions f, g, F^* are

analytic of the parameter μ such that $f, g, F^* \bmod \mu^l$ is completely determined by $F \bmod \mu^l$ for each $l \geq 0$.

By Theorem 1.4 and Lemma 3.7, a normalization ϕ of M_μ and the operation of $\bmod \mu^l$ are commutative, i.e., the following two results give the same information up to $\bmod \mu^l$ for each l :

$$\left\{ \begin{array}{l} \phi(M_\mu) \\ \phi(M_\mu \bmod \mu^l) \end{array} \right. \bmod \mu^l.$$

Let $N_\sigma = \varphi \circ E \circ \psi$ be a normalization of M and $M' \equiv \psi(M)$. Then M' depends analytically on the parameter a in the initial value $\sigma = (C, a, \rho, r)$. By Lemma 3.7 we can compute the normalization E up to a given order of a inclusive.

Lemma 3.8. *Let k be a positive integer ≥ 7 . If M is an analytic real hypersurface in normal form such that*

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, u) + \sum_{\min(s,t) \geq 2, s+t \geq k+1} F_{st}(z, \bar{z}, u),$$

where

$$\sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, u) \neq 0.$$

Then there is a vector $a \in \mathbb{C}^n$ and a normalization N_σ , $\sigma = (id_{n \times n}, a, 1, 0)$, such that

$$N_\sigma(M) : v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2}^{s+t=k-1} F_{st}^*(z, \bar{z}, u) + \sum_{\min(s,t) \geq 2, s+t \geq k} F_{st}^*(z, \bar{z}, u),$$

where

$$\sum_{\min(s,t) \geq 2, s+t=k-1} F_{st}^*(z, \bar{z}, u) \neq 0.$$

Proof. The real hypersurface M is defined as follows:

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2, s+t=k}^{s+t=k+1} F_{st}(z, \bar{z}, u) + \sum_{s+t \geq k+2} O(z^s \bar{z}^t).$$

Let M' be a real hypersurface obtained after the mapping ψ in the decomposition

$$N_\sigma = \varphi \circ E \circ \psi$$

where $\phi_\sigma = \varphi \circ \psi$ is the local automorphism of a real hyperquadric with initial value $\sigma \in H$.

Then the real hypersurface M' depends analytically on the parameter a in $\sigma = (C, a, \rho, r)$. By Lemma 3.7 we can compute the normalization E up to $O(|a|^2)$.

By the mapping ψ in the decomposition of $N_\sigma = \varphi \circ E \circ \psi$, we obtain

$$\begin{aligned}
v &= \langle z, z \rangle \\
&+ \sum_{\min(s,t) \geq 2, s+t=k}^{s+t=k+1} \sum_{\alpha} u \left\{ a^\alpha \left(\frac{\partial F_{st}}{\partial z^\alpha} \right) (z, \bar{z}, u) + \bar{a}^\alpha \left(\frac{\partial F_{st}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, u) \right\} \\
&+ \sum_{\min(s,t) \geq 2, s+t=k} \sum_{\alpha} i \langle z, z \rangle \left\{ a^\alpha \left(\frac{\partial F_{st}}{\partial z^\alpha} \right) (z, \bar{z}, u) - \bar{a}^\alpha \left(\frac{\partial F_{st}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, u) \right\} \\
&+ \sum_{\min(s,t) \geq 2, s+t=k} \sum_{\alpha} i (\langle z, a \rangle - \langle a, z \rangle) \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \\
&+ \sum_{\min(s,t) \geq 2, s+t=k}^{s+t=k} (1 - 2i(1-s)\langle z, a \rangle + 2i(1-t)\langle a, z \rangle) F_{st}(z, \bar{z}, u) \\
&+ \sum_{\min(s,t) \geq 2, s+t=k+2} O(z^s \bar{z}^t) + O(|a|^2).
\end{aligned}$$

For $k \geq 8$, by the normalization E up to $O(|a|^2)$, we obtain

$$\begin{aligned}
v &= \langle z, z \rangle \\
&+ \sum_{\min(s+1,t) \geq 2, s+t=k-1} \sum_{\alpha} u a^\alpha \left(\frac{\partial F_{s+1,t}}{\partial z^\alpha} \right) (z, \bar{z}, u) \\
&+ \sum_{\min(s,t+1) \geq 2, s+t=k-1} \sum_{\alpha} u \bar{a}^\alpha \left(\frac{\partial F_{s,t+1}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, u) \\
&+ \sum_{\min(s,t) \geq 2, s+t=k} F_{st}(z, \bar{z}, u) + \sum_{\min(s,t) \geq 2, s+t=k+1} O(z^s \bar{z}^t) + O(|a|^2).
\end{aligned}$$

For $k = 7$, the condition $\Delta^3 F_{33} = 0$ can be achieved without any effects on the following term up to $O(|a|^2)$:

$$F_{42}^*(z, \bar{z}, u) = \sum_{\alpha} \left\{ a^\alpha \left(\frac{\partial F_{52}}{\partial z^\alpha} \right) (z, \bar{z}, u) + \bar{a}^\alpha \left(\frac{\partial F_{43}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, u) \right\} + O(|a|^2).$$

By the normalization E up to $O(|a|^2)$, we obtain

$$\begin{aligned}
v &= \langle z, z \rangle \\
&+ \sum_{\min(s+1,t) \geq 2, s+t=6} \sum_{\alpha} u a^\alpha \left(\frac{\partial F_{s+1,t}}{\partial z^\alpha} \right) (z, \bar{z}, u) \\
&+ \sum_{\min(s,t+1) \geq 2, s+t=6} \sum_{\alpha} u \bar{a}^\alpha \left(\frac{\partial F_{s,t+1}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, u) \\
&+ \kappa(u) \langle z, z \rangle^3 + \sum_{\min(s,t) \geq 2, s+t=7} F_{st}(z, \bar{z}, u) \\
&+ \sum_{\min(s,t) \geq 2, s+t=8} O(z^s \bar{z}^t) + O(|a|^2)
\end{aligned}$$

where $\kappa(u)$ is determined so that $\Delta^3 F_{33}^* = 0$ up to $O(|a|^2)$.

Since $N_\sigma(M)$ depends analytically on the parameter a in $\sigma = (C, a, \rho, r)$ (cf. [Pa2]), we obtain for sufficiently small $a \in \mathbb{C}^n$

$$\sum_{s+t=k-1} F_{st}^*(z, \bar{z}, u) \neq 0.$$

This completes the proof. \square

Theorem 3.9. *Let M be a nonspherical nondegenerate analytic real hypersurface. Then, for $\dim M = 3$, there exists $\sigma \in H$ such that*

$$N_\sigma(M) : v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2, s+t \geq 6} F_{st}(z, \bar{z}, u),$$

where

$$F_{24}(z, \bar{z}, u) \neq 0.$$

For $\dim M \geq 5$, there exists $\sigma \in H$ such that

$$N_\sigma(M) : v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u),$$

where

$$F_{22}(z, \bar{z}, u) \neq 0.$$

We present a proof of Theorem 3.9 in the paper [Pa3]. Notice that the case of $\dim M = 3$ is an immediate consequence of Lemma 3.8.

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